THE FIRST WHITEHEAD LEMMA FOR MALCEV ALGEBRAS¹

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ABSTRACT. In the present paper the first Whitehead lemma for separable Malcev algebras is proved, the dimensions being finite and the characteristic of the base field necessarily equal to zero. A consequence is the theorem of Malcev-Harish-Chandra for Malcev algebras. To get the lemma a structure theorem for modules over semisimple split Malcev algebras is proved.

1. Introduction. In this paper Whitehead's first lemma is proved for separable Malcev algebras. It says that every derivation of a separable Malcev algebra in a Malcev module is inner, the characteristic of the base field being zero. A consequence is the theorem of Malcev-Harish-Chandra for Malcev algebras.

These theorems are known for Lie algebras which are special Malcev algebras, for alternative, and for Jordan algebras. The lemma is proved by means of the classification of irreducible Malcev modules over simple split Malcev algebras. This classification is done in [1].

2. Definitions. Let A be an algebra over a field k , the characteristic of k not equal to 2. A is a *Malcev algebra* if

$$(1) \ x^2 = 0,$$

$$(2) \ (xy \cdot z)t + (yz \cdot t)x + (zt \cdot x)y + (tx \cdot y)z = ty \cdot xz \text{ is satisfied for all } x, y, z, t \in A.$$

If $R: A \rightarrow \text{End}(A)$ is the (right) regular representation of A with $R(x): y \mapsto yx$, then by (2) we have equivalently

$$(3) \ R(xy \cdot z) = R(x)R(y)R(z) - R(z)R(x)R(y) - R(yz)R(x) + R(y)R(zx).$$

A *Malcev (bi-)module* over a Malcev algebra A over k is a k -module M with two bilinear compositions $A \times M \rightarrow M$ and $M \times A \rightarrow M$ so that $xm = -mx$ and (2) is valid for all $x, y, z \in A$ and $t \in M$. The *semidirect sum* $B = A + M$ of A and a Malcev module M is a Malcev algebra, B being the direct sum of the vectorspaces A and M , with the multiplication $(x + m) \cdot (x' + m') := xx' + xm' + mx'$ if $x, x' \in A, m, m' \in M$. R denotes the regular representation of B with $R(u): v \mapsto vu$ for $u, v \in B$.

A bimodule M over A is a *Lie module* if $xm = -mx$ and $m \cdot xy = mx \cdot y$

Received by the editors August 14, 1974.

AMS (MOS) subject classifications (1970). Primary 17A30, 17E05; Secondary 17B05, 17B15, 17D05.

Key words and phrases. Malcev algebras, Malcev modules, derivations, Whitehead's first lemma, theorem of Malcev-Harish-Chandra, Lie algebras.

¹ This paper is part of a dissertation submitted to the University of Hamburg, prepared under Professor Hel Braun.

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$-my \cdot x$ for $m \in M$ and $x, y \in A$. Set $J(u, v, w) := uv \cdot w + vw \cdot u + wu \cdot v$, $u, v, w \in B$; B as afore. The maximal Lie submodule $N_M := \{m \in M | J(m, x, y) = 0 \text{ for all } x, y \in A\}$ of a Malcev module M over A is the *module nucleus*. A derivation D of A in M is a k -linear mapping $D: A \rightarrow M$ with $(xy)D = (x)Dy + x(y)D$ if $x, y \in A$. D is an *inner derivation* if D can be extended to an inner derivation of B . The k -module of the inner derivations of A in M is generated by $\partial(x, m)$ and $\iota(m')$, $x \in A$, $m \in M$, and $m' \in N_M$, which are the restrictions from B onto A of the inner derivations $D(x, m) := (R(x), R(m)) + R(xm)$ and $R(m')$ of B , $(, ,)$ being the usual commutator product. Let ρ be the canonic representation of A in M with $\rho(x): M \rightarrow M$, $m \mapsto mx$, $x \in A$. In the subsequent, A denotes a Malcev algebra, M a Malcev module over A , both taken finite dimensional if nothing different is stated.

Let $\varphi: A \rightarrow k[x]$, $k[x]$ the ring of the polynomials in x over k . φ is called a *primary function* if $\varphi(a)$ is irreducible for every $a \in A$, and $M_\varphi := \{m \in M | (m)(\varphi(a)(\rho(a)))^r = 0, r = r(a) \in \mathbb{N}, a \in A\}$ is the *primary component* of M relative to φ , φ is *essential* over M if $M_\varphi \neq \{0\}$, and equally for M_φ . $d(p)$ denotes the degree of $p \in k[x]$. If $d(\varphi(a)) = 1$ for all $a \in A$ we have $M_\alpha = M_\varphi$, M_α the *root space* corresponding to the *root* $\alpha: A \rightarrow k$, $a \mapsto \alpha(a)$ with $\varphi(a) = x - \alpha(a)$. For nilpotent A , M has a *primary decomposition* over A into primary components, $M = \bigoplus M_{\pi_i}$, $i = 1, \dots, n$, $n \in \mathbb{N}$ [3]. $\{\pi_i(a) | M_{\pi_i} \neq \{0\}, i = 1, \dots, n\}$ is the set of the irreducible factors of the minimum polynomial of $\rho(a)$. A is *splitting* over M if the characteristic roots of each $\rho(a)$ are in k . Then the primary decomposition is a *root space decomposition*.

LEMMA 1. Let A and M be as above and H a nilpotent subalgebra, splitting over A and M . One has

$$(4) A_\alpha A_\beta \subset A_{\alpha+\beta} \text{ for } \alpha \neq \beta, A_\alpha^2 \subset A_{2\alpha} + A_{-\alpha},$$

$$(5) J(A_\alpha, A_\beta, A) = \{0\} \text{ if } \alpha \neq \beta \neq \gamma \neq \alpha.$$

If M is irreducible and non-Lie over A , we have

$$(6) M_\alpha A_\alpha \subset M_{-\alpha} \text{ and } M_\alpha A_\beta = \{0\} \text{ if } \alpha \neq 0 \text{ and } \beta \neq 0, \alpha, -\alpha,$$

$$(7) M_\alpha \neq \{0\} \Rightarrow A_\alpha \neq \{0\} \text{ or } A_{-\alpha} \neq \{0\}.$$

If the characteristic of k is 0 and H is an abelian Cartan subalgebra, then

$$(8) M_0 H = \{0\} \text{ [1], [3].}$$

3. **Theorems.** A is *split* if A contains a splitting Cartan subalgebra. The seven-dimensional simple split non-Lie Malcev algebra is denoted *algebra of type* C_M^- , and the regular module over A *module of type* C_M^- . The two-dimensional irreducible non-Lie Malcev module over the three-dimensional Lie algebra of type A_1 is called *module of type* M_2 .

THEOREM 1. Let A be a semisimple split Malcev algebra over a field of characteristic 0, M a Malcev module over A , $A = \bigoplus A_i$, $i = 1, \dots, n$, $n \in \mathbb{N}$, the decomposition into the direct sum of simple algebras. Then we have $M = \bigoplus M_r$, $r = 1, \dots, m$, $m \in \mathbb{N}$, with the following properties: M_1 is the maximal Lie submodule of M . For each $r > 1$ there exists i , $1 \leq i \leq n$, so that M_r is non-Lie and of type M_2 or C_M^- over A_i and $M_r A_j = \{0\}$ if $j \neq i$ with $1 \leq j \leq n$. If M is of type M_2 , A_i is a Lie algebra of type A_1 , and in the second case A_i is of type C_M^- .

PROOF. Since M is completely reducible over A [3, Corollary 8], we can

assume that M is irreducible and not Lie. For semisimple A the Cartan subalgebras are the self-normalizing nilpotent subalgebras [3].

Let H be a splitting Cartan subalgebra of the split algebra A . Then $H_i := H \cap A_i$ is a splitting Cartan subalgebra of A_i , $i = 1, \dots, n$, and $H = \bigoplus H_i$. Let γ_i be a root of H_i in A_i , and $\gamma_i^*: H \rightarrow k$ the linear form defined by

$$\gamma_i^* := \begin{cases} \gamma_i(h) & \text{if } h \in H_i, \\ 0 & \text{if } h \in H_j, j \neq i. \end{cases}$$

Then the essential roots of H in A are the extensions γ_i^* of the essential roots γ_i of each H_i in A_i . One has $A_\beta = (A_i)_{\gamma_i}$ if $\beta = \gamma_i^* \neq 0$.

Let Δ denote the set of the essential roots of H in A , Δ_i the set of the essential roots of H_i in A_i so that $\Delta = \bigcup \Delta_i$, $\Delta_i^* := \{\gamma^* | \gamma \in \Delta_i\}$, $i = 1, \dots, n$. If H is splitting over M , we get from (7) that $M = \bigoplus M_\beta$, $\beta \in \Delta$, noting that $\beta \in \Delta$ implies $-\beta \in \Delta$. M is not a zero module, which means $AM \neq \{0\}$. Thus we get by (4) and (8) that there exists a weight $\gamma^* \in \Delta$, $\gamma^* \neq 0$, and $\gamma^* \in \Delta_i^*$ for a particular i . Set

$$M' := \sum (M_{\beta^*} + M_{\beta^*} A_{-\beta^*}), \quad \beta \in \Delta_i, \beta \neq 0, \quad M' = M'(i).$$

We have $M' \neq \{0\}$, and $M'A_i \subset M'$ by (4) and (6). For $j \neq i$ we get

$$M'A_j = \sum (M_{\beta^*} A_j + M_{\beta^*} A_{-\beta^*} \cdot A_j), \quad \beta \in \Delta_i, \beta \neq 0.$$

Now

$$M_{\beta^*} A_j = M_{\beta^*} H_j + M_{\beta^*} (A_j)_1,$$

$(A_j)_1 := \bigoplus A_{\gamma^*}$, $\delta \in \Delta_j$, $\delta \neq 0$ (the Fitting one component of A_j relative to H_j). Thus we get $M_{\beta^*} A_j = \{0\}$, for the first term in the sum is the zero space by (8) since $\beta^*(h) = 0$ if $h \in H_j$, and the second is zero by (6). On the other hand we get

$$J(M_{\beta^*}, A_{-\beta^*}, A_j) = J(M_{\beta^*}, A_{-\beta^*}, H_j) + J(M_{\beta^*}, A_{-\beta^*}, (A_j)_1) = \{0\} \quad \text{by (5).}$$

Therefore

$$M_{\beta^*} A_{-\beta^*} \cdot A_j \subset M_{\beta^*} A_j \cdot A_{-\beta^*} + M_{\beta^*} \cdot A_{-\beta^*} A_j = \{0\},$$

since $M_{\beta^*} A_j = \{0\}$ and $A_i A_j = \{0\}$. Thus M' is a nonzero submodule of M , hence $M' = M$, and we have $MA_j = \{0\}$ if $j \neq i$. M is an irreducible non-Lie Malcev module over the simple Malcev algebra A_i . There exist exactly two cases [1]: Either A_i is the three-dimensional Lie algebra of type A_1 , and M is a module of type M_2 over A_i , or A_i is of type C_m^- , and M is the regular module.

It remains to show that if H is splitting over A , then it is splitting over M . Assume H is not splitting over M . Let $M = \bigoplus M_{\pi_i}$, $i = 1, \dots, m$, be the decomposition of M into essential primary components. Let $\bar{A}, \bar{M}, \bar{H}$ be the base field extensions of A, M, H corresponding to the base field extension of k to an algebraic closure $\bar{k}$. $\bar{H}$ is a Cartan subalgebra of the Malcev algebra $\bar{A}$ [3],

splitting over $\bar{A}$ and $\bar{M}$. $\bar{A}_i$ is of the same type as A_i . For if A_i is a Lie algebra, its Dynkin diagram is invariant by a base field extension. If A_i is of type $C_{\bar{M}}$, a standard basis of A_i is a standard basis of $\bar{A}_i$, thus the type remains unchanged [5].

$\bar{M}$ is completely reducible over $\bar{A}$. We have $(\bar{A}_{\bar{\beta}}) = (\bar{A})_{\bar{\beta}}$, $\beta \in \Delta$, $\bar{\beta}$ the linear extension of β to $\bar{H}$, by the linearity of the essential roots if $\text{char } k = 0$. We get $(\bar{M}_{\bar{\pi}}) = \bigoplus (\bar{M}_{\bar{\pi}})_{\gamma_{ij}}$, $j = 1, \dots, m = m(i) \in \mathbb{N}$, $\gamma_{ij}: \bar{H} \rightarrow \bar{k}$ an essential root of $\bar{H}$ over $\bar{M}$. Every root space M_{β} generates a submodule M' over A with H splitting over M' . This is obvious by (4), considering the canonic imbedding in $\bar{M}$. Thus for any i we have $\max \{d(\pi_i(h)) | h \in H\} \geq 2$. By (7) there exists an essential root $\gamma = \gamma_{ij}$ of $\bar{H}$ over $\bar{M}$, $\gamma = \bar{\beta}$, $\beta \in \Delta_i^*$. For this i and any $h \in H$ we have $x - \gamma(h)|\pi_i(h)$. From $\gamma(h) = \beta(h) \in k$ and the irreducibility of $\pi_i(h)$ we obtain $d(\pi_i(h)) = 1$, which is a contradiction. H is splitting over M . $\square$

A Malcev algebra is *separable* if every base field extension of the algebra is semisimple. We now get

THEOREM 2. (*The first Whitehead lemma for Malcev algebras.*) *Let A be a separable Malcev algebra over a field of characteristic 0, and M a Malcev module over A . Let D be a derivation of A in M . Then D is an inner derivation.*

PROOF. Let A be split and semisimple. Since M is completely reducible over A we can restrict ourselves to the case of irreducible M . Let M be a zero module, $MA = \{0\}$. It is well known and trivial that D is inner, for $(A)D = (A^2)D \subset (A)DA \subset MA = \{0\}$ so that $D = \iota(0)$.

If A is simple and non-Lie, A is of type $C_{\bar{M}}$. Either M is the zero module or regular [1]. Since every derivation of A is inner [5, p. 455], D is an inner derivation.

Let A be semisimple and M an A Lie module. If A is a Lie algebra then it is well known that all Lie derivations are inner. Otherwise we have $A = CE$ where C is a Lie algebra and E a direct sum of algebras of type $C_{\bar{M}}$. M is a Lie module over E , so we have by [1] that $ME = \{0\}$ and, therefore, $(E)D = \{0\}$. The restrictions of D onto C and E are derivations of C and E into M . So there exists $m \in M$ with $(x)D = xm$ for all $x \in C$. If $y \in E$ then one has $ym = 0$; thus $D = \mathbf{r}(m)$.

Let M be non-Lie. From Theorem 1 we get that there exists an i so that either A_i is of type $C_{\bar{M}}$, and M regular over A_i , and thus the restriction of D onto A_i is inner, or A_i is of type A_1 and M of type M_2 . Then the restriction of D onto A_i is inner by [1]. Thus we have $D = D' := \sum \partial(x_r, m_r)$ on A_i , $r = 1, \dots, s$, $s \in \mathbb{N}$ with $x_r \in A_i$ and $m_r \in M$. Since $MA_j = \{0\}$ if $j \neq i$, it follows that $D = D'$.

Let A be separable, and $\tilde{k}$ be an extension of k so that the corresponding base field extension $\tilde{A}$ of A is split. Let $\tilde{M}$ denote the base field extension of M . D defines an inner derivation $\tilde{D}$ of $\tilde{A}$ in $\tilde{M}$ which extends D . As is known, it follows that already D is inner. $\square$

COROLLARY 1. *Let A and M be as in the theorem, $D: A \rightarrow M$ a derivation. Then*

$$D = \sum \partial(x_i, m_i), \quad i = 1, \dots, r, \quad r \in \mathbb{N}, \quad x_i \in A, \quad m_i \in M.$$

PROOF. Because N_M is completely reducible over A , we can assume for a derivation $\mathfrak{h}(m)$, $m = \sum x_i m_i$, $x_i \in A_i$, $m_i \in N_M$, $i = 1, \dots, s$, $s \in \mathbb{N}$. For $m \in N_M$ and all $x, y \in A$ we get

$$(x)\mathfrak{h}(ym) = (x)R(ym) = (x)(R(y), R(m)) = (x)\partial(\frac{1}{2}y, m). \quad \square$$

COROLLARY 2. *Let A be a Malcev algebra and C a separable subalgebra of A , the characteristic of k being 0. Then every derivation of C in A can be extended to an inner derivation of A .*

For two ideals I and B of A define the ideal operator K_B by

$$K_B(I) := IB + IB \cdot A \cdot K_B(I)$$

is an ideal of A . Set $K_B^0(I) := I$, $K_B^n := K_B K_B^{n-1}$, $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ with $K_I^n(I) = \{0\}$, I is K -nilpotent. We have (no condition on the dimension)

LEMMA 2. *Let I be an ideal of the Malcev algebra A , $\text{char } k \neq 2$. I is nilpotent exactly if I is K -nilpotent.*

PROOF. 1. $I^{2n} \subset K_I^n(I)$. For by (3) each product of $2n$ elements of I is a linear combination of right normal products in $r \geq n+1$ elements out of I .

2. $K_I^{2n}(I) \subset I^{n+1}$ follows by induction on n . Assuming it for $n \geq 0$ we get by (2),

$$K_I^{2(n+1)}(I) = K_I^2(K_I^{2n}(I)) \subset K_I^{2n}(I)I \subset I^{n+2}. \quad \square$$

If A is a Malcev algebra, $A = B \oplus S$, B a semisimple subalgebra, S the solvable radical of A , then the decomposition is a *Wedderburn* or *Levi decomposition* and B a *Wedderburn* or *Levi factor* of A . If U and V are subalgebras, f an automorphism of the algebra, and $(U)f = V$, U and V are *conjugate*. An *inner automorphism* of A is a product of automorphisms $\exp(D)$ where D is a nilpotent inner derivation of A .

We then get further:

THEOREM 3. (*Theorem of Malcev-Harish-Chandra for Malcev algebras.*) *Let A be a Malcev algebra over a field of characteristic 0, and $A = B \oplus S$ a Wedderburn decomposition. If C is a separable subalgebra then there exists an inner automorphism f of the algebra A so that $(C)f \subset B$.*

The proof follows by modifying Harish-Chandra's proof (cf. [2]) for Lie algebras. To this aim one has to substitute the ordinary solvability and nilpotency by L -solvability and K -nilpotency which reduce the defining conditions to conditions on ideals, and to observe Corollary 1.

We have again:

COROLLARY 3. *If A has a Wedderburn decomposition then every separable subalgebra is a subalgebra of a Wedderburn factor.*

COROLLARY 4. *Let S be the radical of A , A/S separable. Then any two Wedderburn factors are conjugate by an inner automorphism.*

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