LOCAL RESOLVENTS OF OPERATORS WITH ONE-DIMENSIONAL SELF-COMMUTATOR

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ABSTRACT. Let $T = H + iJ$ be an irreducible operator on a Hilbert space with one-dimensional self-commutator. It is known that the selfadjoint operator H is absolutely continuous. Let E_H denote the absolutely continuous support of H . In this note the following theorem is proven:

THEOREM. *If there exists a real number p such that $\operatorname{ess\,inf} E_H < p < \operatorname{ess\,sup} E_H$ and $\int_{E_H} |t - p|^{-1} dt < \infty$, then the operator T has a nontrivial invariant subspace.*

Let $\mathcal{H}$ be a separable Hilbert space with inner product $(\cdot, \cdot)$. Let T be a bounded linear operator on $\mathcal{H}$. The operator T is called hyponormal in case its self-commutator $T^*T - TT^* = D$ is nonnegative. If the adjoint T^* is a hyponormal operator, then T is called cohyponormal. If the self-commutator $T^*T - TT^*$ is a one-dimensional operator, then either T or T^* is hyponormal.

It is not known at present whether every operator with a one-dimensional self-commutator has a nontrivial invariant subspace. In this note a result is described which increases the class of operators with one-dimensional self-commutator that are known to have nontrivial invariant subspaces.

Let T be a cohyponormal operator. The cartesian decomposition of T will be expressed $T = H + iJ$, where H and J are selfadjoint. For the purposes of this note it can be assumed that T is an irreducible operator. In this case the selfadjoint operators H and J are absolutely continuous [10, Theorem 3.2.1]. Suppose $H = \int t dG_t$ is the spectral resolution of H . Then there is a Borel set E_H in the real line, determined up to a set of measure zero with the property $\int_{E_H} dG_t = I$ and if $\int_F dG_t = I$, for $F \subset E_H$, then $E_H \setminus F$ is of Lebesgue measure zero. This set E_H (really E_H is an equivalence class of Borel sets) will be called the absolutely continuous support of H .

It is known [10, Theorem 3.4.1] that if $T = H + iJ$ is cohyponormal, then $\sigma(H)$ (the spectrum of H) is the projection on the x -axis of $\sigma(T)$. Similarly, $\sigma(J)$ is the projection of $\sigma(T)$ onto the y -axis. An operator with disconnected spectrum is known to have nontrivial invariant subspaces. It follows that every cohyponormal operator $T = H + iJ$, where $\sigma(H)$ or $\sigma(J)$ is not an interval, has a nontrivial invariant subspace.

The main result in this note is the following:

THEOREM 1. *Let $T = H + iJ$ be an irreducible operator with one-dimensional self-commutator. Let E_H be the absolutely continuous support of H and assume*

Received by the editors December 13, 1974 and, in revised form, February 19, 1975.
 AMS (MOS) subject classifications (1970). Primary 47B20; Secondary 47A15.

$\sigma(H) = [a, b]$. If for some $p \in (a, b)$, $\int_{E_H} |t - p|^{-1} dt < \infty$, then the operator T has a nontrivial invariant subspace.

The proof of Theorem 1 will be based on some estimates of the local resolvent of the vector in the range of $T^*T - TT^*$. These estimates are deduced from results in Carey and Pincus [1]. The fact that local resolvents can be used to obtain invariant subspaces for hyponormal operators is in Stampfli [12]. In the case where T is cohyponormal with no point spectrum $TT^* - T^*T = \langle \cdot, \varphi \rangle \varphi$, then there is a weakly continuous extension of $(T - \lambda)^{-1} \varphi$ onto the whole complex plane $\mathbb{C}$ (see Putnam [11]). In the final section of this note it is shown that this extension of $(T - \lambda)^{-1} \varphi$ is not an analytic extension across the boundary of $\sigma(T)$.

1. Preliminaries and notations. From now on it will be assumed that $T = H + iJ$ is an irreducible cohyponormal operator with one-dimensional self-commutator. The notation T_λ will be used for $T - \lambda I$. It is convenient to normalize T such that $TT^* - T^*T = (2/\pi)(\cdot, \varphi)\varphi$, where φ is a unit vector in $\mathcal{H}$.

J. D. Pincus [7] has provided a useful unitary invariant for the operator T . There is a bounded measurable function g defined on the plane which satisfies

$$1 + (\pi i)^{-1}((J - \omega)^{-1}(H - z)^{-1}\varphi, \varphi) \\ = \exp \left\{ \frac{1}{2\pi i} \iint g(x + iy) \frac{dx dy}{(x - z)(y - \omega)} \right\},$$

where $\text{Im } z \neq 0$ and $\text{Im } \omega \neq 0$. The function g is called the principal function of T and has the following properties:

- (i) The function g vanishes off $E_H \times E_J$ and satisfies $0 \leq g \leq 1$ (see [1]).
- (ii) The spectrum of T consists of all complex λ such that every neighborhood of λ intersects the set $\{z: g(z) \neq 0\}$ in a set of positive measure (see [4] and [8]).
- (iii) The essential spectrum of T consists of all complex λ such that every neighborhood of λ intersects the sets $\{z: g(z) \neq 0\}$ and $\{z: g(z) \neq 1\}$ in a set of positive measure [1].
- (iv) The point spectrum of T consists of all complex $\lambda = \mu + i\nu$ such that

$$\iint_{B(\lambda)} \frac{1 - g(x + iy)}{(x - \mu)^2 + (y - \nu)^2} dx dy < \infty;$$

here, $B(\lambda) = \{z: |z - \lambda| < 1\}$, [1].

Let $\lambda = \mu + i\nu$ be a complex number. It is useful to introduce the function

$$(1) \quad \nu_g(\lambda) = \frac{1}{\pi} \iint \frac{g(x + iy)}{(x - \mu)^2 + (y - \nu)^2} dx dy.$$

For our purposes we will need the following estimate which follows easily from the results in [1] (see, in particular the proof of Theorem 6 of [1]).

(v) Let λ be complex and let $T_\lambda^* T_\lambda = \int t dE_t$, $T_\lambda T_\lambda^* = \int t dF_t$ be the spectral resolutions of $T_\lambda^* T_\lambda$ and $T_\lambda T_\lambda^*$, respectively. Then

$$1 + \frac{2}{\pi} \int t^{-1} d(E_t \varphi, \varphi) = \exp[v_g(\lambda)],$$

$$1 - \frac{2}{\pi} \int t^{-1} d(F_t \varphi, \varphi) = \exp[-v_g(\lambda)].$$

In case the operator H has simple spectrum, then Dao-Xeng Xa [13] has shown that T is unitarily equivalent to an operator S acting on $L^2(E_H)$ of the form:

$$(2) \quad Sf(s) = sf(s) + i \left[a(s)f(s) + \frac{b(s)}{\pi i} \int_{E_H}^* \frac{\bar{b}(t)f(t)}{t-s} dt \right];$$

where, a, b are bounded measurable functions on E_H , $b(t) \neq 0$ a.e. and a is real valued. The singular integral is interpreted as a Cauchy principal value, that is, $\int^* = \lim_{\epsilon \rightarrow 0} \int_{|t-s| \geq \epsilon}$. vector valued generalization of the above result of Dao-Xeng Xa is in [7]. In fact, the theory of unitary invariants for operators with trace class self-commutator has been significantly developed [3], [6].

In order to have concrete examples of the determining function we observe that the determining function for the operator S defined by (2) is the function

$$g(x + iy) = \frac{\arg}{\pi} \left[\frac{a(x) + |b(x)|^2 - y}{a(x) - |b(x)|^2 - y} \right];$$

where, $0 \leq \arg \leq 2\pi$.

Let A be an operator on $\mathcal{H}$ which has no point spectrum. For $x \in \mathcal{H}$, the function $x(\lambda) = (A - \lambda)^{-1}x$ has a maximal $\mathcal{H}$ -valued analytic continuation from the resolvent set of A . The domain of this maximal analytic continuation is called the local (analytic) resolvent set of x and is denoted by $\rho_A(x)$. The complementary set is called the local spectrum of x and will be denoted by $\sigma_A(x)$. If σ is a compact subset in the plane, then it is easy to see that $M(\sigma) = \{x \in \mathcal{H} : \sigma_A(x) \subset \sigma\}$ is a linear manifold in $\mathcal{H}$ which is invariant under the operator A . Stampfli [12] has shown that $M(\sigma)$ is sometimes a closed subspace when A is hyponormal. More precisely:

THEOREM 2. *Let A be a hyponormal operator such that A^* has no point spectrum. If σ is a compact subset of the plane, then $M(\sigma)$ is a closed subspace invariant under the operator A .*

Unfortunately, there are hyponormal operators such that $M(\sigma) = \mathcal{H}$ or $M(\sigma) = (0)$ for every compact set σ . One such example is the unilateral shift. On the other hand there are plenty of examples where the subspace $M(\sigma)$ is a nontrivial invariant subspace for a hyponormal operator (see, e.g. [9] and [12]). Our proof of Theorem 1 will provide more examples where $M(\sigma)$ is nontrivial.

It is interesting to compare the problem of local analytic spectra in the hyponormal case with the cohyponormal case. If A is a cohyponormal operator with empty point spectrum, then there exists a vector $x \in \mathcal{H}$ such that $\sigma_A(x)$ is a nonempty proper subset of $\sigma(A)$ [12], however, no analogue of Theorem 2 is known in the cohyponormal case.

If A is an operator on $\mathcal{H}$ which is one-to-one, and $AA^* = \int t dG_t$ is the spectral resolution of AA^* , then the vector $x \in \mathcal{H}$ belongs to the domain of

A^{-1} if and only if $\int t^{-1} d(G_t x, x) < \infty$. In this case the norm of $A^{-1}x$ verifies $\|A^{-1}x\|^2 = \int t^{-1} d(G_t x, x)$. Property (v) describes when φ is in the domain of T_λ^{-1} or $((T_\lambda)^*)^{-1}$.

An irreducible hyponormal operator has no point spectrum. It follows that φ is in the domain of $((T_\lambda)^*)^{-1}$ if and only if $\nu_g(\lambda) < \infty$. It should be observed that $\nu_g(\lambda) = \infty$ a.e. on the set $\{z: g(z) \neq 0\}$. The fact that the characteristic function of $E_H \times E_J$ dominates the function g implies that the condition

$$(3) \quad \int_{E_H} |t - p|^{-1} dt < \infty$$

is sufficient for φ to be in the domain of $((T_\lambda)^*)^{-1}$, for all λ such that $\operatorname{Re} \lambda = p$. In fact condition (3) implies $\|((T_\lambda)^*)^{-1}\varphi\|$ is bounded on $\operatorname{Re} \lambda = p$.

Let E be a bounded measurable set. Define for real p the function $\nu_E(p) = \int_E |t - p|^{-1} dt$. This function ν_E is infinite at a.e. point in E . On the other hand $\nu_E(p)$ is often finite for points $p \notin E$, for example, if E is closed, then $\nu_E(p) < \infty$, for $p \notin E$.

There are a couple of examples which are interesting in connection with the function ν_E . There is a set $E_L \subset I = [0, 1]$ of positive measure such that $I \setminus E_L$ is a closed nowhere dense set of positive measure and $\nu_{E_L}(p) = \infty$, for all $p \in I \setminus E_L$. This last example is due to Lusin (see, e.g. [2]). On the other hand, there is another set E_0 of positive measure in I such that $I \setminus E_0$ is a closed nowhere dense set of positive measure with $\nu_{E_0}(p) < \infty$, for a.e. $p \in E_0$ (see [2]).

The second equality in property (v) implies that φ is in the domain of T_λ^{-1} , whenever, λ is not in the point spectrum of T ; moreover, $\|T_\lambda^{-1}\varphi\| \leq 1$. This last result has a generalization in Putnam [11].

2. Proof of Theorem 1. We can assume that both operators T and T^* have empty point spectrum. It is clear from the discussion in the preceding section that condition (3) implies $((T_\lambda)^*)^{-1}\varphi$ exists as a bounded $\mathcal{H}$ -valued function on $\operatorname{Re} \lambda = p$. An argument similar to that given in Putnam [11] can be used to show that the function $\varphi(\lambda) = (T^* - \lambda)^{-1}\varphi$ is weakly continuous on $\operatorname{Re} \lambda = p$.

Let C be a circle that contains $\sigma(T^*)$ in its interior and let γ be the open subinterval of $\operatorname{Re} \lambda = p$ which is contained inside the circle C . Form the two closed contours $\Gamma_1 = \gamma \cup [C \cap [\operatorname{Re} \lambda \geq p]]$ and $\Gamma_2 = \gamma \cup [C \cap [\operatorname{Re} \lambda \leq p]]$. We assume Γ_1, Γ_2 are oriented in the counter-clockwise direction. One of the vectors $\varphi_i = \int_{\Gamma_i} \varphi(\lambda) d\lambda$, $i = 1, 2$, is nonzero; here, the integral is defined as a weak integral.

For convenience we assume $\varphi_1 \neq 0$. It is routine to prove that $\varphi_1(\mu) = \int_{\Gamma_1} \varphi(\lambda)(\lambda - \mu)^{-1} d\lambda$ is an analytic continuation of $\varphi_1(\mu) = (T^* - \mu)^{-1}\varphi_1$ onto the domain exterior to Γ_1 . This means that $\sigma_{T^*}(\varphi_1)$ is a proper subset of $\sigma(T^*)$. It follows from Theorem 2 that $M(\sigma_{T^*}(\varphi_1))$ is a nontrivial subspace invariant under T^* . This completes the proof.

3. Conclusion. It is interesting to compare the result in Theorem 1 with the invariant subspace result which follows from property (iv) of the determining function. For definiteness we work with a specific example. Let E be a bounded measurable subset of the real line and consider the singular integral

operator T_E defined on $L^2(E)$ by

$$T_E f(s) = sf(s) + \frac{1}{\pi} \int_E^* \frac{f(t)}{t-s} dt.$$

Of course, T_E is one of the cohyponormal operators defined in equation (2). The principal function is the characteristic function of $E \times [-1, 1]$.

The operator T_E has an eigenvalue if and only if for some real p , the function $\nu_{E^c}(p) < \infty$; here, E^c is the complement of E relative to some bounded interval containing E in its interior. Theorem 1 gives a nontrivial invariant subspace of T_E when $\nu_E(p) < \infty$, for some p satisfying $\text{ess inf } E < p < \text{ess sup } E$.

The proof of Theorem 1 shows that the operator T_F has plenty of nontrivial invariant subspaces whenever $F \subset E_L$. It does not give a similar statement about T_{E_L} ($\nu_{E_L}(p) = \infty$ on $(0, 1)$), however, T_{E_L} has plenty of eigenvalues.

It was shown above that in the absence of point spectrum of T , then $\varphi(\lambda) = (T - \lambda)^{-1}\varphi$ admits a weakly continuous extension (still denoted by $\varphi(\lambda)$) from the resolvent set onto all of $\mathbb{C}$. It develops that $\varphi(\lambda)$ is not analytic at points in the boundary of $\sigma(T)$.

From the fact that $\nu_g(\lambda) = \infty$ a.e. on $\{z: g(z) \neq 0\}$, it follows that $\|\varphi(\lambda)\| = 1$ for points in every neighborhood of a point λ_0 in the boundary of $\sigma(T)$. On the other hand $\|\varphi(\lambda)\| < 1$, for $\lambda \notin \sigma(T)$. This implies that $\varphi(\lambda)$ is not analytic in any neighborhood of λ_0 (see, e.g. Dunford and Schwartz [5, p. 220]).

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