ON THE BOUNDEDNESS OF PSEUDO DIFFERENTIAL OPERATORS IN THE CLASS $L^m_{\rho,1}$

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ABSTRACT. We prove that every pseudo differential operator in the class $L^m_{\rho,1}$, $0 \leq \rho \leq 1$, is bounded in $L^2(\mathbf{R}^n)$ if and only if $m < n(\rho - 1)/2$.

According to Hörmander [3], we denote by $S^m_{\rho,\delta}$, $0 \leq \rho \leq \delta \leq 1$, the class of the symbols $a(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ which satisfy the inequalities:

$$(1) \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq c_{\alpha,\beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}$$

for all $x, \xi \in \mathbf{R}^n \times \mathbf{R}^n$ and every multiorder α, β . We write $L^m_{\rho,\delta}$ for the class of operators A of the form

$$(2) \quad Au(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} a(x, \xi) \hat{u}(\xi) d\xi$$

with $a(x, \xi) \in S^m_{\rho,\delta}$.

A combination of the results of Hörmander [4] and the method in Calderón and Vaillancourt [1] gives

THEOREM 1. *Suppose $\delta < 1$. Every $A \in L^m_{\rho,\delta}$ is bounded in $L^2(\mathbf{R}^n)$ if and only if*

$$(3) \quad m \leq n(\rho - \delta)/2.$$

Theorem 1 fails if $\delta = 1$, $m = 0$, as has been shown in Ching [2] by means of an example of an operator $A \in L^0_{1,1}$ which is not bounded. Our purpose is to study in detail the case $\delta = 1$. We shall prove

THEOREM 1'. *Every $A \in L^m_{\rho,1}$ is bounded in $L^2(\mathbf{R}^n)$ if and only if*

$$(3') \quad m < n(\rho - 1)/2.$$

Actually in the proof of the sufficiency of condition (3') we shall use an argument which requires only that inequalities (1) are satisfied for $\alpha = 0$, $|\beta| \leq n$. To verify the necessity of condition (3') we shall give an example of an operator in $L^m_{\rho,1}$, $m = n(\rho - 1)/2$, which is not bounded in $L^2(\mathbf{R}^n)$; for $m = 0$, $\rho = 1$, we shall obtain essentially the example in Ching [2].

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PROOF OF THEOREM 1'. *Necessity of condition (3')*. Take $\chi \in C_0^\infty(\mathbf{R}^n)$ with $\chi(\xi) = 0$ when $|\xi| \geq \frac{1}{3}$ and $\chi(\xi) = 1$ when $|\xi| \leq \frac{1}{4}$. We define, for $\eta \in \mathbf{R}^n$, $|\eta| \geq 10$,

$$(4) \quad a_\eta(x, \xi) = \sum_{|g| \leq \frac{1}{4}|\eta|^{1-\rho}} e^{-i\langle x, \eta + g|\eta|^\rho \rangle} \chi((\xi - \eta)|\eta|^{-\rho} - g)$$

where g runs over the set of all points in $\mathbf{R}^n$ with integral coordinates. The terms in the sum have disjoint supports and direct computation shows that

$$(5) \quad |D_x^\alpha D_\xi^\beta a_\eta(x, \xi)| \leq c_{\alpha, \beta} (1 + |\xi|)^{|\alpha| - \rho|\beta|}$$

where $c_{\alpha, \beta}$ does not depend on η .

Now let η_j be a sequence in $\mathbf{R}^n$ such that $|\eta_j| = 10^j$. We define

$$(6) \quad a(x, \xi) = \sum_{j=1}^{\infty} b_j |\eta_j|^m a_{\eta_j}(x, \xi)$$

where b_j is a bounded sequence of complex numbers such that $\sum_{j=1}^{\infty} |b_j|^2$ is divergent. Since the functions $a_{\eta_j}(x, \xi)$ have disjoint supports, in view of (5) we can conclude that $a(x, \xi) \in S_{\rho, 1}^m$. We shall prove that the operator $A \in L_{\rho, 1}^m$ defined by (2) is not bounded in $L^2(\mathbf{R}^n)$ if $m = n(\rho - 1)/2$.

Assume the contrary that, for some constant C ,

$$(7) \quad \|Au\|^2 \leq C\|u\|^2$$

for all $u \in \mathcal{S}$ and test the boundedness of A in the following way. Choose $0 \neq \varphi \in \mathcal{S}$ with $\hat{\varphi}(\xi) = 0$ when $|\xi| \geq \frac{1}{4}$ and set

$$(8) \quad \hat{u}_M(\xi) = \sum_{j=1}^M c_j \sum_{|g| \leq \frac{1}{4}|\eta_j|^{1-\rho}} \hat{\varphi}(\xi - \eta_j - g|\eta_j|^\rho).$$

We see that

$$a(x, \xi) \hat{u}_M(\xi) = \sum_{j=1}^M c_j b_j |\eta_j|^m \sum_{|g| \leq \frac{1}{4}|\eta_j|^{1-\rho}} e^{-i\langle x, \eta_j + g|\eta_j|^\rho \rangle} \hat{\varphi}(\xi - \eta_j - g|\eta_j|^\rho).$$

If N_j is the number of the points g with $|g| \leq \frac{1}{4}|\eta_j|^{1-\rho}$, we have

$$Au_M = \sum_{j=1}^M c_j b_j |\eta_j|^m N_j \varphi.$$

On the other hand, $\|u_M\|^2 = \sum_{j=1}^M |c_j|^2 N_j \|\varphi\|^2$, since all terms in (8) have disjoint supports. Then, if we apply (7) to u_M , we see that

$$\left(\sum_{j=1}^M c_j b_j |\eta_j|^m N_j \right)^2 \leq C \sum_{j=1}^M |c_j|^2 N_j$$

which implies

$$(9) \quad \sum_{j=1}^M |b_j|^2 |\eta_j|^{2m} N_j \leq C.$$

Since $N_j |\eta_j|^{n(\rho-1)}$ converges to the volume of the unit ball when $j \rightarrow \infty$, the assumption $m = n(\rho - 1)/2$ implies $\sum_{j=1}^{\infty} |b_j|^2 < \infty$. But this contradicts our hypothesis. The proof of the necessity is complete.

Sufficiency of condition (3'). Let $a(x, \xi)$ be in $S^m_{\rho,1}$, $m < n(\rho - 1)/2$. We can assume $a(x, \xi) = 0$ when, in $\xi = (\xi_1, \dots, \xi_n)$, $\xi_h = 0$ for some h , $h = 1, \dots, n$ (we have no loss of generality, as it is easy to verify).

It will be sufficient to prove the boundedness of A^* , adjoint of A . We have

$$(10) \quad \widehat{A^* u}(\xi) = \int e^{-i\langle x, \xi \rangle} \overline{a(x, \xi)} u(x) dx$$

and thus

$$|\widehat{A^* u}(\xi)|^2 = \iint e^{i\langle x-y, \xi \rangle} a(x, \xi) \overline{a(y, \xi)} \overline{u(x)} u(y) dx dy.$$

We can write

$$(11) \quad a(x, \xi) \overline{a(y, \xi)} = \int_0^{\xi_1} \dots \int_0^{\xi_n} \frac{\partial^n [a(x, \vartheta) \overline{a(y, \vartheta)}]}{\partial \vartheta_1 \dots \partial \vartheta_n} d\vartheta$$

and

$$\frac{\partial^n [a(x, \vartheta) \overline{a(y, \vartheta)}]}{\partial \vartheta_1 \dots \partial \vartheta_n} = \sum \partial_{j_1} \dots \partial_{j_k} a(x, \vartheta) \partial_{j_{k+1}} \dots \partial_{j_n} \overline{a(y, \vartheta)}$$

where $\partial_{j_1} \dots \partial_{j_k} = \partial^k / \partial \vartheta_{j_1} \dots \partial \vartheta_{j_k}$. In the sum, $\{j_1, \dots, j_k\}$ runs over all the subsets of $\{1, \dots, n\}$ and $\{j_1, \dots, j_k\} \cup \{j_{k+1}, \dots, j_n\} = \{1, \dots, n\}$. Then we have

$$|\widehat{A^* u}(\xi)|^2 \leq \sum \int \left| \int e^{i\langle x-y, \xi \rangle} \partial_{j_1} \dots \partial_{j_k} a(x, \vartheta) \partial_{j_{k+1}} \dots \partial_{j_n} \overline{a(y, \vartheta)} \overline{u(x)} u(y) dx dy \right| d\vartheta.$$

Using the inequality $2|pq| \leq |p|^2 + |q|^2$, we obtain

$$\begin{aligned} 2 \left| \int e^{i\langle x-y, \xi \rangle} \partial_{j_1} \dots \partial_{j_k} a(x, \vartheta) \partial_{j_{k+1}} \dots \partial_{j_n} \overline{a(y, \vartheta)} \overline{u(x)} u(y) dx dy \right| \\ \leq (1 + |\vartheta|)^{\sigma_k} \left| \int e^{i\langle x, \xi \rangle} \partial_{j_1} \dots \partial_{j_k} a(x, \vartheta) \overline{u(x)} dx \right|^2 \\ + (1 + |\vartheta|)^{-\sigma_k} \left| \int e^{-i\langle y, \xi \rangle} \partial_{j_{k+1}} \dots \partial_{j_n} \overline{a(y, \vartheta)} u(y) dy \right|^2 \end{aligned}$$

where the real constants σ_k , $k = 0, 1, \dots, n$, will be fixed later. Then we have

$$\begin{aligned} 2 \int |\widehat{A^* u}(\xi)|^2 d\xi \\ \leq \sum \int (1 + |\vartheta|)^{\sigma_k} \left| \int e^{i\langle x, \xi \rangle} \partial_{j_1} \dots \partial_{j_k} a(x, \vartheta) \overline{u(x)} dx \right|^2 d\xi d\vartheta \\ + \sum \int (1 + |\vartheta|)^{-\sigma_k} \left| \int e^{-i\langle y, \xi \rangle} \partial_{j_{k+1}} \dots \partial_{j_n} \overline{a(y, \vartheta)} u(y) dy \right|^2 d\xi d\vartheta. \end{aligned}$$

Assuming at first $\sigma_k = -\sigma_{n-k}$ and using Parseval's formula, we obtain

$$(12) \quad \|A^* u\|^2 \leq \int |u(x)|^2 \left\{ \sum \int (1 + |\vartheta|)^{\sigma_k} |\partial_{j_1} \dots \partial_{j_k} a(x, \vartheta)|^2 d\vartheta \right\} dx.$$

Now, if (3') is satisfied, we can choose σ_k such that $\sigma_k + 2m - 2k\rho < -n$, that is, since $\sigma_k = -\sigma_{n-k}$,

$$n + 2m - 2(n - k)\rho < \sigma_k < -n - 2m + 2k\rho.$$

Since

$$\int (1 + |\vartheta|)^{\sigma_k} |\partial_{j_1} \dots \partial_{j_k} a(x, \vartheta)|^2 d\vartheta \leq C_1 \int (1 + |\vartheta|)^{\sigma_k + 2m - 2k\rho} d\vartheta < \infty$$

we have $\|A^* u\|^2 \leq C_2 \|u\|^2$. The proof of Theorem 1' is complete.

In [1] Calderón and Vaillancourt have studied the boundedness of pseudo differential operators of the form

$$(13) \quad Bu(x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi \rangle} b(x, y, \xi) u(y) dy d\xi.$$

We say that B in (13) is of order m and type ρ , δ_1 , δ_2 , $0 \leq \rho \leq \delta_1 \leq 1$, $0 \leq \rho \leq \delta_2 \leq 1$, if the symbol $b(x, y, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$ satisfies the inequalities

$$(14) \quad |D_x^\alpha D_y^\beta D_\xi^\gamma b(x, y, \xi)| \leq c_{\alpha, \beta, \gamma} (1 + |\xi|)^{m - \rho|\gamma| + |\alpha|\delta_1 + |\beta|\delta_2}$$

for all $x, y, \xi \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ and every multiorder α, β, γ . Assume for simplicity $\delta_1 = \delta_2 = \delta$; the results of Calderón and Vaillancourt [1] and Hörmander [4] give for operators of form (13) the following

THEOREM 2. *Suppose $\delta < 1$. Every pseudo differential operator of order m and type ρ, δ, δ is bounded in $L^2(\mathbf{R}^n)$ if and only if*

$$(15) \quad m \leq n(\rho - \delta).$$

We now want to study the case $\delta = 1$.

THEOREM 2'. *Every pseudo differential operator of order m and type $\rho, 1, 1$ is bounded in $L^2(\mathbf{R}^n)$ if and only if*

$$(15') \quad m < n(\rho - 1).$$

PROOF OF THEOREM 2'. We observe that a repetition of the first part of the proof in Calderón and Vaillancourt [1] easily gives the sufficiency of condition (15').

To verify the necessity we have to show that if $m = n(\rho - 1)$ there exists an operator of order m and type $\rho, 1, 1$ which is not bounded in $L^2(\mathbf{R}^n)$. In view of Theorem 1', we can take $a(x, \xi) \in S_{\rho, 1}^{m/2}$, with $m/2 = n(\rho - 1)/2$, such that the corresponding operator A in (2) is not bounded. Consider the operator $B = AA^*$. If we write it in the form (13), its symbol $b(x, y, \xi) = a(x, \xi)a(y, \xi)$ satisfies on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ inequalities (14), with $m = n(\rho - 1)$ and $\delta_1 = \delta_2 = 1$. Since B cannot be bounded, the proof of Theorem 2' is complete.

Finally, we observe that the sufficiency of condition (3') in Theorem 1' can be also easily deduced from Theorem 2'. In fact, the boundedness of $A \in L^m_{\rho,1}$ in (2) is equivalent to the boundedness of $B = AA^*$, which is an operator of type $\rho, 1, 1$ and order $2m$. This alternative argument has been pointed out to us by the referee; it requires that inequality (1) be satisfied only for $\alpha = 0$ and $|\beta| \leq k$ with $2k > n$, which is less restrictive than $k = n$, the condition we need in our proof.

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