POINTWISE INDUCED OPERATORS ON L_p -SPACES

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ABSTRACT. In this note we present a characterization of pointwise induced operators on L_p -spaces with finite measures, $1 \leq p < \infty$. An operator P is pointwise induced if and only if $|Pu| = P|u|$ and $P1 = 1$. As an application we obtain a characterization of the linear positive isometries mapping 1 into 1.

1. Introduction. Let (X_i, Σ_i, m_i) , $i = 1, 2$, be positive measure spaces with finite measures. A function $\varphi: X_2 \rightarrow X_1$ is called a *nonsingular transformation* when it is (Σ_2, Σ_1) -measurable and $m_2 \varphi^{-1}(A) = 0$ whenever $m_1(A) = 0$. For each such transformation we define the operator P^φ from Σ_1 -measurable complex (real) functions on X_1 to Σ_2 -measurable complex (real) functions on X_2 by setting $P^\varphi u = u\varphi$. From the nonsingularity of φ it follows that $P^\varphi u = P^\varphi u'$ a.e. whenever $u = u'$ a.e.; therefore we can (and will) identify functions equal almost everywhere. For any $1 \leq p \leq \infty$ the operator P^φ restricted to $L_p(X_1)$ will be denoted by P_p^φ .

In the sequel we answer the following two questions:

- (1) Under what conditions on φ does the operator P^φ take $L_p(X_1)$ into $L_p(X_2)$?
- (2) Under what conditions on a linear operator $P: L_p(X_1) \rightarrow L_r(X_2)$ does there exist $\varphi: X_2 \rightarrow X_1$ such that $P = P_p^\varphi$?

The former is answered in §2, the latter in §3. In §4 we give an application of our main result, the answer to (2).

The functional spaces $L_p(X_i)$ can be considered either complex or real. The characteristic function of A is denoted by 1_A ; the function identically equal one is denoted by 1.

2. Properties of P^φ . The following proposition answers the first question.

PROPOSITION 1. *Let $\varphi: X_2 \rightarrow X_1$ be a nonsingular transformation and let $1 \leq p < \infty$. The operator P^φ takes $L_p(X_1)$ into $L_p(X_2)$ if and only if there exists a constant K such that $m_2 \varphi^{-1} \leq Km_1$. Moreover, if k is the infimum of all such constants, then $\|P_p^\varphi\| = k^{1/p}$.*

PROOF. If $m_2 \varphi^{-1} \leq Km_1$ then for any $u \in L_p(X_1)$

$$\int |P^\varphi u|^p dm_2 = \int |u|^p dm_2 \varphi^{-1} \leq K \int |u|^p dm_1 = K \|u\|^p;$$

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therefore, $P^\varphi u \in L_p(X_2)$ and $\|P^\varphi u\| \leq K^{1/p} \|u\|$. Taking the infimum over all such K we obtain $\|P^\varphi u\| \leq k^{1/p} \|u\|$. However, for any $A \in \Sigma_1$ we also have

$$m_2 \varphi^{-1}(A) = \|P^\varphi 1_A\|^p \leq \|P^\varphi\|^p \|1_A\|^p = \|P^\varphi\|^p m_1(A).$$

Thus $\|P^\varphi\| \geq k^{1/p}$ and, consequently, the equality holds.

To prove the converse we assume that there exists a sequence of sets $A_n \in \Sigma_1$ such that $m_2 \varphi^{-1}(A_n) > n \cdot m_1(A_n)$ for $n = 1, 2, \dots$. We define the sequence of functions

$$u_n = 1_{A_n} / (n^2 \cdot m_1(A_n))$$

and set $u \in (\sum u_n)^{1/p}$ so that $\int |u|^p dm_1 = \sum 1/n^2 < \infty$ and $\int |P^\varphi u|^p dm_2 = \int |u|^p dm_2 \varphi^{-1} \geq \sum 1/n = \infty$. Therefore, the function $u \in L_p(X_1)$ and $P^\varphi u \notin L_p(X_2)$; the proof of the proposition is complete.

Obviously P^φ maps $L_\infty(X_1)$ into $L_\infty(X_2)$ for any nonsingular transformation φ . It is also easy to see, under the assumptions of the proposition, that the operator P^φ takes $L_r(X_1)$ into $L_p(X_2)$ whenever $1 \leq r \leq p \leq \infty$.

If $P = P^\varphi$ for some nonsingular transformation $\varphi: X_2 \rightarrow X_1$, then, clearly, the following properties hold:

- (i) $|Pu| = P|u|$ for any u in the domain of P , and
- (ii) $P1 = 1$.

We show, under an additional, but not very restrictive assumption about X_1 , that these two properties characterize operators of the type P_p^φ for $1 \leq p < \infty$.

3. Pointwise inducing of P . We first observe that any linear operator $P: L_p(X_1) \rightarrow L_r(X_2)$ satisfying (i) is nonnegative, i.e. carries nonnegative functions into nonnegative functions. In the sequel we use the following general result about nonnegative operators in L_p -spaces; the proof of Lemma 1 for $p = r = 1$ was suggested to us by Professor C. Ryll-Nardzewski. (We note that the finiteness of the measures is not needed in Lemma 1.)

LEMMA 1. *Let $1 \leq p \leq \infty$, $1 \leq r \leq \infty$. If $P: L_p(X_1) \rightarrow L_r(X_2)$ is a nonnegative linear operator then P is continuous.*

PROOF. Let $\alpha_k > 0$ and $\sum \alpha_k < \infty$. If P is not continuous then there exists a sequence $0 \leq u_k \in L_p(X_1)$ such that $\|u_k\|_p \leq 1$ and $\|\sum_{k=1}^n \alpha_k P u_k\|_r \rightarrow \infty$. By the completeness of $L_p(X_1)$ we have $\sum \alpha_k u_k \in L_p(X_1)$. Thus

$$P \sum \alpha_k u_k \geq P \sum_{k=1}^n \alpha_k u_k = \sum_{k=1}^n \alpha_k P u_k$$

yielding the contradiction that $\|P \sum \alpha_k u_k\|_r \geq \|\sum_{k=1}^n \alpha_k P u_k\|_r \rightarrow \infty$.

In the real case the above proof actually yields a more general statement. Namely, it is sufficient to assume that both spaces are B -lattices, i.e. real Banach spaces and vector lattices, satisfying $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ (see [11, p. 369] for the definitions). We also remark that Lemma 1 gives an alternative proof of the "only if" part of Proposition 1.

LEMMA 2. *If a linear operator $P: L_\infty(X_1) \rightarrow L_\infty(X_2)$ satisfies (i) and (ii) then*

P takes characteristic functions into characteristic functions.

PROOF. For any $A \in \Sigma_1$ we have $|1_A \pm 1_{A^c}| = 1$ so that

$$1 = P1 = P|1_A \pm 1_{A^c}| = |P1_A \pm P1_{A^c}|;$$

consequently $P1_A(x) = 0$ or 1 and $P1_A$ must be a characteristic function.

R. R. Phelps proved [8] that for any linear operator $P: L_\infty(X_1) \rightarrow L_\infty(X_2)$ belonging to the set

$$\mathbf{P} = \{P: \|P\| \leq 1 \text{ and } 0 \leq P1 \leq 1\}$$

the following conditions are equivalent:

- (iii) P takes characteristic functions into characteristic functions,
- (iv) P is multiplicative,
- (v) P is an extreme point of $\mathbf{P}$.

The following proposition is another characterization of $\text{ext } \mathbf{P}$.

PROPOSITION 2. For any $P \in \mathbf{P}$ condition (iii) is equivalent to the following condition:

- (vi) $P = 1_A Q$ for some $A \in \Sigma_2$ and Q satisfying (i) and (ii).

PROOF. (iii) $\Rightarrow$ (vi). We have $P1 = 1_A$ for some $A \in \Sigma_2$. Define $Q: L_\infty(X_1) \rightarrow L_\infty(X_2)$ by $Qu = Pu + 1_{A^c}u$. Clearly, the operator Q satisfies (ii), (iii), and $P = 1_A Q$. To prove (i) we assume, without loss of generality, that u is a simple function. Thus, there exist $\alpha_i \in \mathbb{C}$ and disjoint sets $A_i \in \Sigma_1$, $i = 1, \dots, n$, such that $u = \sum \alpha_i 1_{A_i}$. We observe that if B and C are disjoint then $Q1_B$ and $Q1_C$, as well as $Q(1_B + 1_C)$, are characteristic functions forcing the supports of $Q1_B$ and $Q1_C$ to be disjoint. Thus

$$|Qu| = |\sum \alpha_i Q1_{A_i}| = \sum |\alpha_i| Q1_{A_i} = Q \sum |\alpha_i| 1_{A_i} = Q|u|.$$

- (vi) $\Rightarrow$ (iii) is an immediate consequence of Lemma 2.

In the light of Lemma 1 we have the following characterization of a subset of $\text{ext } \mathbf{P}$.

COROLLARY. For any nonnegative $P: L_\infty(X_1) \rightarrow L_\infty(X_2)$ conditions (i) $\wedge$ (ii) and (iii) $\wedge$ (ii) are equivalent.

A. J. Ellis showed [3] that if X_i are compact Hausdorff spaces, $C(X_i)$ their algebras of real-valued continuous functions, and $P: C(X_1) \rightarrow C(X_2)$ a non-negative linear operator, then (i) $\wedge$ (ii) is equivalent to each of the following three conditions: (vii) P is an extreme point of the set of all positive operators mapping 1 into 1 , (viii) P is multiplicative and maps 1 into 1 , (ix) the adjoint of P maps the extreme points of $\{f \in C(X_2)^*: f \geq 0, f(1) = 1\}$ into extreme points of $\{f \in C(X_1)^*: f \geq 0, f(1) = 1\}$.

Therefore, in the real case, the equivalence in the corollary can be also obtained by comparing, e.g., (iv) with (viii) and by using the well-known fact that there exist compact Hausdorff spaces Y_i such that $L_\infty(X_i)$ is isomorphic with $C(Y_i)$ for $i = 1, 2$ (see e.g., [2, p. 445]).

A more abstract statement of equivalences (iv) $\Leftrightarrow$ (v) and (vii) $\Leftrightarrow$ (viii) has been proved in [4, Theorem 3].

We recall that a topological space is called a *Borel space* if it is homeomorphic to a Borel subset of the Hilbert cube. Whenever a measure structure on a Borel space appears, we assume that the measure is positive, finite, and defined on the Borel σ -algebra.

LEMMA 3. *Let X_1 be a Borel space and $P: L_\infty(X_1) \rightarrow L_\infty(X_2)$ a linear operator satisfying (i) and (ii). If $h_n \downarrow 0$ a.e. implies $Ph_n \rightarrow 0$ a.e. for any sequence $0 \leq h_n \in L_\infty(X_1)$, then there exists a nonsingular transformation $\varphi: X_2 \rightarrow X_1$ such that $P = P_\infty^\varphi$.*

PROOF. From Lemma 2 it follows that P carries characteristic functions into characteristic functions. Let $\mathbf{B}_i$ denote the quotient Boolean σ -algebra of Σ_i modulo sets of measure zero. For every $[A] \in \mathbf{B}_1$ we define $\Phi([A]) = [B] \in \mathbf{B}_2$ when $P1_A = 1_B$. It is easy to see that $\Phi: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ is a well-defined homomorphism of the Boolean algebras. Moreover, if $[A_1] \supseteq [A_2] \supseteq \dots$ and $\bigcap [A_n] = 0$ then $1_{A_n} \downarrow 0$ a.e. so that $1_{B_n} = P1_{A_n} \downarrow 0$ a.e. with $[B_n] = \Phi([A_n])$. Therefore $\bigcap \Phi([A_n]) = 0$ so Φ is a σ -homomorphism. By [9, 5.1(c)] there exists a nonsingular transformation $\varphi: X_2 \rightarrow X_1$ such that $\Phi([A]) = [\varphi^{-1}(A)]$ for all $A \in \Sigma_1$ (see also [10, 32.5] for a simpler proof of that fact). This implies $P1_A = 1_A \varphi$ which, by the continuity of P (Lemma 1), proves our assertion.

We are now in a position to prove our main result.

THEOREM 1. *Let X_1 be a Borel space, let $1 \leq p < \infty$, $1 \leq r \leq \infty$, and let $P: L_p(X_1) \rightarrow L_r(X_2)$ be a linear operator satisfying (i) and (ii). Then $P = P_p^\varphi$ for some nonsingular transformation $\varphi: X_2 \rightarrow X_1$.*

PROOF. From well-known measure theory results it follows that P_∞ , the restriction of P to $L_\infty(X_1)$, satisfies the assumptions of Lemma 3. Hence, there exists a nonsingular transformation $\varphi: X_2 \rightarrow X_1$ such that $P_\infty = P_\infty^\varphi$. From Lemma 1 and the fact that $L_\infty(X_1)$ is dense in $L_p(X_1)$, we can conclude that $P = P_p^\varphi$.

The theorem is no longer true when $p = \infty$. Indeed, if $p = r = \infty$ then every homomorphism of the Boolean quotient σ -algebra $\mathbf{B}_1$ into $\mathbf{B}_2$ induces a linear operator satisfying (i) and (ii) (define the operator on the characteristic functions and extend it to the whole space). On the other hand, any homomorphism inducing P_∞^φ must be a σ -homomorphism.

4. Isometries. In this section we give an application of Theorem 1 and prove a result about isometries of L_∞ -spaces.

The following theorem gives a characterization of the nonnegative linear isometries on L_p -spaces ($1 \leq p < \infty$) taking 1 into 1.

THEOREM 2. *Let X_1 be a Borel space and let $1 \leq p < \infty$. Then an operator $P: L_p(X_1) \rightarrow L_p(X_2)$ is a nonnegative linear isometry (not necessarily onto) with $P1 = 1$ if and only if there exists a measure preserving transformation $\varphi: X_2 \rightarrow X_1$ such that $P = P_p^\varphi$.*

PROOF. If P is a nonnegative isometry then for any $u \in L_p(X_1)$ we have $\|u\| = \|Pu\| \leq \|P|u|\| = \|u\|$ which implies $|Pu| = P|u|$. By Theorem 1 there exists a nonsingular transformation $\varphi: X_2 \rightarrow X_1$ such that $P = P_p^\varphi$. For any

$A \in \Sigma_1$ we have $m_1(A) = \|1_A\|^p = \|P1_A\|^p = \|1_{\varphi^{-1}(A)}\|^p = m_2\varphi^{-1}(A)$, so that φ is measure preserving.

The converse is obvious.

There are several known results about isometries of L_p -spaces; assuming $P \geq 0$ and $P1 = 1$, Theorem 2 is a particular case. However, Theorem 2 is valid when $p = 2$; this is not the case for the other results. In Banach's book [1] we find a characterization of the onto isometries of the real spaces $L_p([0, 1])$ and l_p for $1 \leq p < \infty$, $p \neq 2$. J. Lamperti generalized these results [5]. He gives a characterization of isometries (not necessarily onto) of L_p -spaces with σ -finite measures for $0 < p < \infty$, $p \neq 2$. Our result, for $p \neq 2$ and $(X_1, \Sigma_1, m_1) = (X_2, \Sigma_2, m_2)$, can be deduced from [5, Theorem 3.1] with the already mentioned [9, 5.1(c)]. Lamperti's results have been generalized to the reflexive Orlicz spaces by G. Lumer [6], [7]. Again, the space has been assumed to be different from a Hilbert space.

In the case of L_∞ -spaces we have the following result for the onto isometries.

THEOREM 3. *Let X_1 and X_2 be uncountable Borel spaces and let $P: L_\infty(X_1) \rightarrow L_\infty(X_2)$. The operator P is a nonnegative linear isometry onto $L_\infty(X_2)$ if and only if there exists an invertible nonsingular transformation φ such that $P = P_\varphi^\infty$.*

PROOF. The "if" part is clear. To prove the "only if" part we note that by the Banach-Stone theorem P is a multiplicative operator (see [2, V. 8.11 and V. 8.8]); consequently, P takes characteristic functions into characteristic functions. Therefore, P induces an isomorphism between the quotient Boolean σ -algebras $\mathbf{B}_1$ and $\mathbf{B}_2$ (see the proof of Lemma 3). By [9, 6.2] there exists a transformation φ satisfying the required conditions. When applying [9] we make use of the known fact that there exist uncountable subsets of measure zero in both X_1 and X_2 .

5. A generalization. If the measures m_1 and m_2 are infinite, the condition $P1 = 1$ in Theorem 1 is no longer meaningful. However, if the measures are σ -finite, we can generalize Theorem 1 by substituting (i) $\wedge$ (ii) by the new condition:

(iii) $_\sigma$ P maps integrable characteristic functions into integrable characteristic functions.

Taking an appropriate partition of X_1 and applying Theorem 1 to each of the parts, we can easily obtain the following result for σ -finite measures m_1 and m_2 .

THEOREM 4. *Let X_1 be a Borel space, $1 \leq p < \infty$, $1 \leq r \leq \infty$, and let $P: L_p(X_1) \rightarrow L_r(X_2)$ be a linear operator satisfying (iii) $_\sigma$. Then there exists a set $B \in \Sigma_2$ and a nonsingular transformation $\varphi: B \rightarrow X_1$ such that*

$$Pu(x) = \begin{cases} u\varphi(x) & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

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ADDED IN PROOF. The author has recently learned of a general result (Chapter III, Proposition 9.2 in [S]) containing our implication $(i) \wedge (ii) \Rightarrow (iii)$. The result is valid for arbitrary B -lattices with quasi-interior elements. Our Lemma 1 is stated in [S] in more general form as Theorem 5.3, Chapter II.

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