ON THE FAILURE OF THE WEAK BETH PROPERTY

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ABSTRACT. A general method for showing the failure of the weak Beth definability property for certain pairs of logics is discussed. Applications are made to the game logic, various infinitary logics and partially ordered quantifier logics.

In [5], Craig essentially shows that the satisfaction relation for second order logic is implicitly but not explicitly definable in second order logic—thereby showing that the analogue of Beth's definability theorem fails for second order logic. (See also Kreisel's review [10].) In this note we shall elaborate on Craig's idea to obtain the failure of the Beth property in various other situations. In so doing we shall obtain some new results as well as give some new proofs of old results.

All the logics we deal with are assumed to be such that their formulas are coded by sets in some standard way. In particular, we assume that the variables are indexed by ordinals, that v_α is coded by $(0, \alpha)$ and that each logic contains a binary relation symbol $\bar{E}$ such that atomic formulas of the form $\bar{E}(v_\alpha, v_\beta)$ are coded by $(1, \alpha, \beta)$.

The results we obtain actually show that an even weaker property than the usual Beth property fails. Following Friedman [6], we say that a logic L_1 has the *weak Beth property* with respect to a logic L_2 (Weak Beth (L_1, L_2)) if for every sentence φ of L_1 , of type $\mu \cup \{\bar{R}\}$, such that each structure $\mathfrak{M}$ of type μ has *exactly one* expansion $\langle \mathfrak{M}, R \rangle$ which is a model of φ , there is a formula ψ of L_2 , of type μ , such that

$$\models \varphi \rightarrow (\forall v_1) \cdots (\forall v_n) [\bar{R}(v_1, \dots, v_n) \leftrightarrow \psi(v_1, \dots, v_n)].$$

The usual Beth property, Beth (L_1, L_2) , is obtained from the above by weakening the assumption "exactly one" to "at most one".

Using the method of this paper, it is possible to prove a general theorem asserting the failure of the weak Beth property for a large class of pairs of logics. However to do so would involve some cumbersome exposition which we would rather avoid. Hence we shall restrict ourselves to a detailed proof of the most involved case and then indicate the changes necessary to prove our other results.

We shall assume that the reader is familiar with the infinitary logics $L_{\kappa\lambda}$. The logics $L_{\kappa\lambda}^G$ are obtained from $L_{\kappa\lambda}$ by adding the game quantifier. That is,

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the formation rules of $L_{\kappa\lambda}^G$ are those of $L_{\kappa\lambda}$ with the addition of: if φ_n are formulas of $L_{\kappa\lambda}^G$, and x_n are variables then $(\forall x_0)(\exists x_1)(\forall x_2)(\exists x_3) \cdots \bigwedge_{n \in \omega} \varphi_n(x_0, \dots, x_{2n})$ is a formula of $L_{\kappa\lambda}^G$ provided it has less than λ free variables and x_i , for $i > 2n$, does not occur free in φ_n . ($L_{\infty\omega}^G$ is then what is called $L_{\infty G}$ in [2].) A detailed definition of satisfaction for such logics may be found e.g. in Keisler [9].

THEOREM 1. Weak Beth($L_{\omega_1\omega}^G, L_{\infty\infty}^G$) fails.

PROOF. Let μ be the similarity type consisting of the single binary relation symbol $\bar{E}$, and let φ_1 be the conjunction of

(i) KP, and

(ii) $\sim (\exists v_0)(\exists v_1) \cdots \bigwedge_{n \in \omega} \bar{E}(v_{n+1}, v_n)$,

where KP is the Kripke-Platek set of axioms for admissible sets (see Barwise [2]). Note that φ_1 is a sentence of $L_{\omega_1\omega}^G[\mu]$ and that each model of φ_1 is well-founded (and hence isomorphic to a transitive admissible set).

Let $\langle A, E \rangle$ be a model of φ_1 . Using induction over the well-founded relation E , it is easy to show that there is a unique binary relation R_A on A such that $(x, f) \in R_A$ iff

(0) in $\langle A, E \rangle$, x is a formula of $L_{\infty\infty}^G[\mu]$ and f is a function whose domain is an ordinal

and one of the following inductive clauses holds:

(1) in $\langle A, E \rangle$, $x = (1, \alpha, \beta)$ and $\alpha, \beta \in \text{dom}(f)$ and $f(\alpha) \in f(\beta)$,

(2) in $\langle A, E \rangle$, $x = \sim y$ and all indices of the free variables of y belong to $\text{dom}(f)$ and $(y, f) \notin R_A$,

(3) in $\langle A, E \rangle$, $x = \bigwedge Y$ and for all $y \in Y$, $(y, f) \in R_A$,

(4) in $\langle A, E \rangle$, $x = (\exists Z)y$ where Z is a set of variables and there is a function g such that f and g agree on the indices of the free variables of x , $\text{dom}(g)$ is an ordinal containing the indices of all variables in Z and $(y, g) \in R_A$,

(5) in $\langle A, E \rangle$, $x = (\forall z_0)(\exists z_1)(\forall z_2)(\exists z_3) \cdots \bigwedge_{n \in \omega} y_n$ and if $f^{(v_0, \dots, v_{2n})}$ is the function defined (in $\langle A, E \rangle$) by

$$f^{(v_0, \dots, v_{2n})}(\alpha) = \begin{cases} v_\alpha & \text{if } \alpha \text{ is the index of } z_k \text{ and } 0 \leq k \leq 2n, \\ f(\alpha) & \text{otherwise,} \end{cases}$$

then (in the real world) for every v_0 there is v_1 such that for every v_2 there is $v_3 \dots$ such that for all $n \in \omega$, $(y_n, f^{(v_0, \dots, v_{2n})}) \in R_A$.

Next observe that some simple modifications of the proof of Barwise's Lemma 2.4 in [1] show that all syntactical notions of $L_{\infty\infty}^G[\mu]$ (in the sense of $\langle A, E \rangle$) are uniformly (Δ_1) definable over all models of φ_1 . This definability enables us to express conditions (0)–(4) by formulas of $L_{\omega\omega}[\mu \cup \{\bar{R}\}]$ and condition (5) by a formula of $L_{\omega_1\omega}^G[\mu \cup \{\bar{R}\}]$. Hence, if $\varphi_2(\bar{R})$ is

$$(\forall x)(\forall f)[\bar{R}(x, f) \equiv (0) \wedge ((1) \vee (2) \vee (3) \vee (4) \vee (5))],$$

then $\langle A, E, R \rangle \models \varphi_1 \wedge \varphi_2(\bar{R})$ iff $R = R_A$.

Moreover, if $\langle A, \in, R \rangle$ is a transitive model of $\varphi_1 \wedge \varphi_2(\bar{R})$ and A has the property:

(P) if $a \in A$, $b \subseteq A$ and $\bar{b} \leq a\bar{a}$, then $b \in A$,

then R will be the satisfaction relation for $L_{\infty\infty}^G[\mu] \cap A$ over A , i.e., for $\xi, f \in A$, $(\xi, f) \in R$ iff $\langle A, \in \rangle \models_f \xi$. The proof of this turns on the absoluteness of the syntactical notions (which are absolute because they are Δ_1) as well as the fact that (P) insures that the g in (4) will exist in A if it exists at all.

Now let $\varphi(\bar{R})$ be $(\varphi_1 \wedge \varphi_2(\bar{R})) \vee (\sim \varphi_1 \wedge (\forall x)(\forall y)\bar{R}(x, y))$. Then $\varphi(\bar{R}) \in L_{\omega_1\omega}^G[\mu \cup \{\bar{R}\}]$, and the above considerations immediately imply that for every structure $\langle A, E \rangle$ there is a unique R such that $\langle A, E, R \rangle \models \varphi(\bar{R})$. Hence the hypothesis of the weak Beth property holds. To complete the proof it suffices to show that the conclusion is not satisfied by any formula ψ of $L_{\kappa\kappa}^G[\mu]$ for any regular κ .

Hence assume that $\psi(v_0, v_1) \in L_{\kappa\kappa}^G[\mu]$ and is such that $\models \varphi(\bar{R}) \rightarrow (\forall v_0)(\forall v_1)[\psi(v_0, v_1) \equiv \bar{R}(v_0, v_1)]$. Then if H_κ is the set of all sets hereditarily of power less than κ , H_κ is a transitive model of φ_1 and has property (P).

Hence

$$\langle H_\kappa, \in, R \rangle \models \varphi(\bar{R})$$

and so

$$\langle H_\kappa, \in, R \rangle \models (\forall v_0)(\forall v_1)[\psi(v_0, v_1) \equiv \bar{R}(v_0, v_1)],$$

where R is the satisfaction relation for $L_{\kappa\kappa}^G[\mu]$ over $\langle H_\kappa, \in \rangle$.

The usual Tarski style diagonalization will now complete the proof. Indeed, let $\xi(v_0)$ be $\psi(v_0, \{\langle 0, v_0 \rangle\})$, where it is understood that each occurrence of $\{\langle 0, v_0 \rangle\}$ is replaced by a formula of $L_{\omega\omega}[\mu]$ using the standard definitions of $\{ \}$, $\langle \rangle$, and 0 . Then if $a \in H_\kappa$ is $\sim \xi(v_0)$, it follows that

$$\langle H_\kappa, \in \rangle \models \xi(a) \text{ iff } \langle H_\kappa, \in \rangle \models \psi(a, \{\langle 0, a \rangle\}) \text{ iff } \langle H_\kappa, \in, R \rangle \models \bar{R}(a, \{\langle 0, a \rangle\})$$

$$\text{iff } (a, \{\langle 0, a \rangle\}) \in R \text{ iff } \langle H_\kappa, \in \rangle \models \sim \xi(a).$$

This contradiction completes the proof. Q.E.D.

Next recall that in [13], Malitz has shown that Beth $(L_{\omega_1\omega_1}, L_{\infty\infty})$ fails. His proof, as given, does not show the failure of Weak Beth $(L_{\omega_1\omega_1}, L_{\infty\infty})$, but can be easily modified to do so. If, however, we replace $L_{\infty\infty}^G$ by $L_{\infty\infty}$ everywhere in the proof of Theorem 1, ignore (5) and note that φ_1 is also a sentence of $L_{\omega_1\omega_1}$, we obtain a new proof of this result, i.e.,

THEOREM 2. Weak Beth $(L_{\omega_1\omega_1}, L_{\infty\infty})$ fails.

Continuing in the same vein, we recall that Gregory [8] has shown that Beth $(L_{\kappa^+\omega}, L_{\kappa^+\kappa})$ fails for all regular $\kappa > \omega$ and actually (with the use of a later result of Tait) for all κ such that $\text{cf}(\kappa) > \omega$. Gregory's proof however, unlike the above mentioned proof of Malitz, does not appear capable of modification to show the failure of the corresponding weak Beth property. Our method, however, does do this, and indeed provides an even stronger result. To show this we first need some definitions.

DEFINITION. An ordinal α is called $L_{\kappa\lambda}$ accessible if there is a sentence $\varphi(\bar{U}, \bar{L}, \dots)$ of $L_{\kappa\lambda}$ such that

(I) if $\langle A, U, <, \dots \rangle \models \varphi$, then $<$ well-orders U , and

(II) there is an $\langle A, U, <, \dots \rangle \models \varphi$ in which $\langle U, < \rangle$ has order type greater than or equal to α . Such a sentence is sometimes said to *pin down* α .

DEFINITION. If A is a transitive set, $\text{ord}(A)$ is the supremum of the ordinals in A .

THEOREM 3.

- (a) Weak Beth($L_{\kappa^+\omega}, L_{\lambda\lambda}$) fails for all regular λ which are $L_{\kappa^+\omega}$ accessible.
- (b) Weak Beth($L_{\kappa^+\omega}, L_{\infty\omega} \cap A$) fails for all admissible A such that $\text{ord}(A)$ is $L_{\kappa^+\omega}$ accessible.

PROOF. (a) Replace (ii) in the proof of Theorem 1 by

(ii') $\varphi_0(U, <, \dots) \wedge (\forall v_0)(\forall v_1)[\bar{E}(v_0, v_1) \rightarrow \bar{F}(v_0) < \bar{F}(v_1)]$ where φ_0 is an $L_{\kappa^+\omega}$ sentence which pins down λ , $\bar{E}$ is the binary relation symbol of (i) and $\bar{F}$ is a unary function symbol. We let μ be the similarity type containing all the symbols of (i) and (ii').

If φ_1 is the conjunction of (i) and (ii'), then each structure $\langle A, E, U, <, F, \dots \rangle$ which is a model of φ_1 is such that $\langle A, E \rangle$ is well-founded. Furthermore, $\langle H_\lambda, \in \rangle$ can be expanded to a model of φ_1 . Since λ is regular, H_λ has the property (P), so that the rest of the proof now follows the pattern of the proof of Theorem 1—again ignoring clause (5).

Part (b) is proven in the same way as part (a) except that the property (P) is not necessary here. Q.E.D.

Our promised strengthening of Gregory's result now follows as an easy corollary of Theorem 3(a). We remark that it was also obtained by Friedman (unpublished) using a very complicated argument.

COROLLARY 3. Weak Beth ($L_{\kappa^+\omega}, L_{\kappa^+\kappa^+}$) fails for all κ with $\text{cf}(\kappa) > \omega$.

PROOF. If $\text{cf}(\kappa) > \omega$ then κ^+ is $L_{\kappa^+\omega}$ accessible as is shown in [3] or [4]. Q.E.D.

In connection with Corollary 3 we should mention that Gregory [8] has also shown that Beth ($L_{\kappa^+\omega}, L_{\infty\kappa}$) fails for all regular $\kappa > \omega$. The failure of the weak Beth property for this situation does not seem susceptible to our method and is still open.¹

The proofs of Theorems 2 and 3 are simple modifications of the ideas in the proof of Theorem 1. We remark that another easy modification of these ideas would show that Weak Beth ($L_{\kappa\omega}, L_{2^{\kappa^+}\kappa}$) fails for all regular κ , if there were an $L_{\kappa\omega}$ sentence φ_0 such that (I) and (II') there is a structure $\langle A, U, <, \dots \rangle$ which is a model of φ_0 in which $\langle U, < \rangle$ has order type greater than or equal to $(2^\kappa)^+$. However, since Beth ($L_{\kappa\omega}, L_{(2^\kappa)^+\kappa}$) holds (see Malitz [13]) it follows that there can be no such sentence φ_0 , so that we have another proof of the result of Lopez-Escobar [11]:

The class of all well-orderings is not PC_Δ in any logic $L_{\kappa\omega}$.

Finally, we shall apply our methods to the simplest of the partially ordered quantifier logics introduced by Henkin, i.e., the extension of $L_{\omega\omega}$ by the quantifier

$$\left(\begin{array}{l} \forall x \exists y \\ \forall z \exists w \end{array} \right).$$

¹ However, S. Shelah announced that Weak Beth ($L_{\kappa^+\omega}, L_{\infty\lambda^+}$) fails when $\kappa = 2^{\lambda^+}$ (see Notices Amer. Math. Soc. 22 (1975). Abstract A-474).

We shall denote this logic by L^H and refer the reader to [12] for a detailed description. In [12] it is shown that L^H fails to have the interpolation property. Here we prove

THEOREM 4. Weak Beth (L^H, L^H) fails.

PROOF. Ehrenfeucht has shown that there is a sentence φ_0 of L^H all of whose models are isomorphic to $\langle \omega, < \rangle$ (see [12]). Using this, it is easy to write down a sentence $\varphi_1(\bar{E})$ of L^H , all of whose models are isomorphic to $\langle H_\omega, \in \rangle$. Now proceed to define inductively the relation R_A , with clauses (3) and (4) limited to finite conjunctions and quantifiers and clause (5) replaced by

(5') in $\langle A, E \rangle$,

$$x = \left(\begin{array}{c} \forall z_0 \exists z_1 \\ \forall z_2 \exists z_3 \end{array} \right) y$$

and if $f^{(v_0, v_1, v_2, v_3)}$ is the function defined by

$$f^{(v_0, v_1, v_2, v_3)}(\alpha) = \begin{cases} v_k & \text{if } \alpha \text{ is the index of } z_k \text{ and } 0 \leq k \leq 4, \\ f(\alpha) & \text{otherwise,} \end{cases}$$

then (in real world) for every v_0 there is v_1 and for every v_2 there is v_3 depending only on v_2 such that $(y, f^{(v_0, v_1, v_2, v_3)}) \in R_A$.

The rest of the proof is now essentially the same as the proof of Theorem 1.

In closing, we would like to add that further applications of the method of this paper can be found in [7].

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