BANACH SPACES WITH BIHOLOMORPHICALLY EQUIVALENT UNIT BALLS ARE ISOMORPHIC

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ABSTRACT. It is shown that in every complex Banach space E with open unit ball $D \subset E$ there is a closed $\mathbb{C}$ -linear subspace $V \subset E$ such that $V \cap D$ is the orbit of the origin $0 \in E$ under the group $\text{Aut}(D)$ of all biholomorphic automorphisms of D . In particular two complex Banach spaces are isometrically equivalent if and only if their open unit balls are biholomorphically equivalent.

Suppose E, F are complex Banach spaces and $U \subset E$ is an open subset. Then a mapping $f: U \rightarrow F$ is called holomorphic if the Fréchet derivative of f at x (denoted by $f'(x)$) exists for every $x \in U$. (For details compare [6].) In this paper we show that the Banach space E is completely determined by the holomorphic structure of its (open) unit ball; more precisely,

THEOREM. *The complex Banach spaces E and $\tilde{E}$ are isometrically isomorphic if and only if there is a biholomorphic mapping from the unit ball of E onto the unit ball of $\tilde{E}$.*

This answers a question L. A. Harris raised in [3, p. 15].

For the proof let E be a fixed complex Banach space with unit ball $D \subset E$. Let $\mathfrak{v}$ denote the set of all holomorphic vector fields on D i.e. of all expressions $X = f(z)\partial/\partial z$ where $f: D \rightarrow E$ is a holomorphic mapping and z denotes the coordinate of D . Then $\mathfrak{v}$ is a complex Lie algebra in the product

$$\left[f(z) \frac{\partial}{\partial z}, g(z) \frac{\partial}{\partial z} \right] = (g'(z)f(z) - f'(z)g(z)) \frac{\partial}{\partial z}.$$

For every integer ν the Lie algebra $\mathfrak{v}$ contains the linear subspace

$$\mathfrak{P}_\nu = \left\{ h(z) \frac{\partial}{\partial z} \mid h: E \rightarrow E \text{ is a homogeneous polynomial of degree } \nu + 1 \right\}.$$

Obviously $[\mathfrak{P}_\nu, \mathfrak{P}_\mu] \subset \mathfrak{P}_{\nu+\mu}$ for every ν and μ .

Since D is a bounded domain in E , the set $\mathfrak{g}$ of all complete holomorphic vector fields on D has the following properties (compare [7]):

(0) Every $h(z)\partial/\partial z \in \mathfrak{g}$ is uniquely determined by $h(0)$ and $h'(0)$ (Cartan's uniqueness theorem).

Received by the editors October 29, 1975.

AMS (MOS) subject classifications (1970). Primary 32M10, 46B99; Secondary 22E65.

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- (1) $\mathfrak{g} \subset \mathfrak{v}$ is a real Lie subalgebra satisfying $\mathfrak{g} \cap i\mathfrak{g} = 0$.
 (2) For every positive $r < 1$, $\|h(z)\partial/\partial z\| = \sup_{\|z\| \leq r} \|h(z)\|$ defines a norm on $\mathfrak{g}$. Any two of these norms are equivalent and make $\mathfrak{g}$ a Banach Lie algebra.

LEMMA 1. *There exists a topological direct sum decomposition*

$$(3) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{P}_0$, $\mathfrak{p} = \mathfrak{g} \cap (\mathfrak{P}_{-1} \oplus \mathfrak{P}_1)$. This decomposition is $\text{ad}(\mathfrak{k})$ -invariant and, furthermore, $\mathfrak{g} \cap \mathfrak{P}_{-1} = \mathfrak{g} \cap \mathfrak{P}_1 = 0$.

PROOF. The holomorphic vector field $Z := i\partial/\partial z$ is in $\mathfrak{g}$ since D is invariant under the corresponding one parameter group $z \mapsto e^{it}z$, $t \in \mathbf{R}$. Consider a holomorphic vector field $X \in \mathfrak{g}$. Then in a certain neighborhood of the origin, X may be given by a convergent series $X = \sum_{\nu=-1}^{\infty} X_{\nu}$, where $X_{\nu} \in \mathfrak{P}_{\nu}$. Since $\mathfrak{g}$ is a real Lie algebra the vector field $Y := p(\text{ad } Z)X = \sum_{\nu=2}^{\infty} p(i\nu)X_{\nu}$ is also in $\mathfrak{g}$, where the real polynomial $p \in \mathbf{R}[t]$ is given by $p = t(t^2 + 1)$. Applying Cartan's uniqueness theorem (0), we get $Y = 0$, i.e. $X_{\nu} = 0$ for $\nu \geq 2$, and hence $X = X_{-1} + X_0 + X_1$. But $X_0 = X + (\text{ad } Z)^2 X$ is also in $\mathfrak{g}$ and the decomposition (3) follows immediately from this. For every $X \in \mathfrak{g} \cap \mathfrak{P}_1$ we have $iX = (\text{ad } Z)X \in \mathfrak{g}$ and, hence $\mathfrak{g} \cap \mathfrak{P}_1 \subset \mathfrak{g} \cap i\mathfrak{g} = 0$. In the same way we get $\mathfrak{g} \cap \mathfrak{P}_{-1} = 0$. Q.E.D.

Next consider the $\mathbf{R}$ -linear subspace $V = \{f(0) | f(z)\partial/\partial z \in \mathfrak{g}\}$ of E . Since $\mathfrak{g} \cap \mathfrak{P}_1 = 0$, for every $\alpha \in V$ there is a unique symmetric continuous bilinear mapping $q_{\alpha}: E \times E \rightarrow E$ such that $(\alpha - q_{\alpha}(z, z))\partial/\partial z \in \mathfrak{g}$.

LEMMA 2. $V \subset E$ is closed.

PROOF. Fix a positive $d \leq \frac{1}{3}$ and consider holomorphic vector fields $X = h(z)\partial/\partial z \in \mathfrak{p}$ satisfying

$$(4) \quad \|h(z)\| \leq d^2/6 \quad \text{if } \|z\| \leq 2d.$$

Denote by $L(E)$ the endomorphism algebra of E and endow the Banach space $F = E \times L(E)$ with the max-norm. Consider the one parameter group $g_t = \exp(tX)$ associated with X and define $u: \mathbf{R} \rightarrow D \times L(E) \subset F$ by $u(t) = (g_t(0), g'_t(0))$. Then there exists a positive $s \leq 1$ such that $u(t) \in U$ for all t with $|t| \leq s$, where $U \subset F$ is the open ball with radius d and center $u(0) = (0, \text{id}_E)$. Then u satisfies the differential equation

$$(5) \quad du/dt = f(u),$$

where $f: D \times L(E) \rightarrow F$ is defined by $f(x, \lambda) = (h(x), h'(x) \circ \lambda)$. It follows from (4) by a simple computation that f' is bounded by 1 on $U \subset D \times L(E)$. Therefore we get

$$(6) \quad \|u(t) - u(0)\| \leq \|f(u(0))\|(e^s - 1) \leq \|h(0)\|3s < sd \quad \text{if } |t| \leq s.$$

(Compare [1, 10.5.1].) In particular we may assume $s = 1$. Applying [2,

Theorem 1] to the function $\delta(g_t - g_t(0))$, where $\delta = (1 + \|g_t(0)\|)^{-1}$, we find a constant $c > 0$ such that for $t \in \mathbf{R}$ and $\|z\| \leq 2d$,

$$(7) \quad \|g_t(z) - z\| \leq c\|u(t) - u(0)\|,$$

i.e.

$$(8) \quad \|g_t(z) - z\| \leq 3c\|h(0)\||t| \quad \text{if } |t| \leq 1, \|z\| \leq 2d.$$

From this we get finally

$$\|X\|_i := \sup_{\|z\| \leq 2d} \|h(z)\| \leq 3c\|h(0)\|,$$

where c does not depend on X . (Compare [7, Korollar 4.6].) Therefore the continuous, $\mathbf{R}$ -linear mapping $\varphi: \mathfrak{p} \rightarrow V$, $\varphi(h(z)\partial/\partial z) = h(0)$ is bijective and has a continuous inverse, i.e. V is complete. Q.E.D.

LEMMA 3. V is a $\mathbf{C}$ -linear subspace of E and every $X \in \mathfrak{g}$ is tangent to V .

PROOF. For every $X = (\alpha - q_\alpha(z, z))\partial/\partial z \in \mathfrak{p}$ and $Y = \lambda(z)\partial/\partial z \in \mathfrak{k}$, we have

$$[X, Y] = (\lambda(\alpha) - \lambda(q_\alpha(z, z)) + 2q_\alpha(\lambda(z), z))\frac{\partial}{\partial z} \in \mathfrak{p},$$

i.e. $\lambda(V) \subset V$ and Y is tangent to V . In particular for $Y = Z = iz\partial/\partial z$, we get $iV = V$ and $q_{i\alpha} = -iq_\alpha$. Since $(\text{ad } X)^2 Z = 4iq_\alpha(\alpha, z) \in \mathfrak{k}$ and $\mathfrak{k}$ tangent to V , we have

$$(9) \quad q_\alpha(\alpha, z) \in V \quad \text{for every } \alpha, z \in V.$$

Substituting $\alpha + z$ for α in (9) we deduce that $q_\alpha(z, z) \in V$ for every $\alpha, z \in V$, i.e. X is tangent to V . Q.E.D.

In [7] it has been proved that the group $G = \text{Aut}(D)$ of all biholomorphic automorphisms of D in a natural way is a real Banach Lie group. The connected component G^0 of G is generated by the image of the exponential map $\exp: \mathfrak{g} \rightarrow G$. Therefore $D \cap V$ is invariant under G^0 by Lemma 3. The mapping $\psi: G^0 \rightarrow D \cap V$ defined by $g \rightarrow g(0)$ is analytic and the differential of ψ at the point $\text{id}_D \in G^0$ is precisely the topological isomorphism $\varphi: \mathfrak{p} \rightarrow V$ of Lemma 2. Therefore, by the implicit function theorem, the orbit $G^0(0)$ of the origin is open in $D \cap V$. Hence, by [5, Lemma 2], we have $G^0(0) = D \cap V$. Let $\Omega = \{x \in D | G^0(x) \text{ is a closed complex submanifold of } D\}$. Obviously $0 \in \Omega$ and Ω is G -invariant since G^0 is normal in G . Let $x \in \Omega$ with $x \neq 0$. Then $W = \{t \in \mathbf{C}: tx \in G^0(x)\}$ is an analytic subset of the disc $\Delta = \{t \in \mathbf{C}: |t| < \|x\|^{-1}\}$.

Since $\{t \in \mathbf{C}: |t| = 1\} \subset W$ we have $W = \Delta$ and therefore $0 \in G^0(x)$, or equivalently, $x \in G^0(0)$. But then

$$\Omega \subset G^0(0) = D \cap V \subset G(0) \subset \Omega, \text{ i.e. } \Omega = G^0(0) = G(0) = D \cap V.$$

PROPOSITION 1. *Suppose D is the unit ball of a complex Banach space E and $G = \text{Aut}(D)$ is the group of all biholomorphic automorphisms of D . Then G is a real Banach Lie group with the following properties:*

(i) *The mapping $G \times D \rightarrow D$ defined by $(g, x) \mapsto g(x)$ is analytic.*

(ii) *$G = G^0 G_0$ where G^0 is the connected component of G and $G_0 = \{g \in G: g(0) = 0\}$ is the isotropy subgroup at the origin.*

There exists a closed $\mathbb{C}$ -linear subspace $V \subset E$ such that

(iii) *$G(0) = D \cap V = \{x \in D: G(x) \text{ is a closed complex submanifold of } D\}$ and $D \cap V$ is a bounded symmetric domain in V on which the groups G and G^0 act transitively.*

PROOF. Statement (i) is contained in [7] and (iii) has already been proved. To prove (ii) let $g \in G$. Then $g(0) \in G(0) = G^0(0)$, i.e. $g(0) = h(0)$ for some $h \in G^0$. But then $h^{-1}g \in G_0$, which proves the statement.

COROLLARY. *Every $g \in G$ admits a holomorphic extension onto an open neighborhood of $\bar{D} \subset E$ (with values in E).*

PROOF. If $g \in G_0$ this is true since $G_0 \subset GL(E)$. If $g \in G^0$ this is true since G^0 is generated by $\exp(\mathfrak{g})$ and since every $X \in \mathfrak{g}$ is holomorphic on E . (Compare [4].)

Now we can prove the Theorem: Suppose $\tilde{E}$ is another complex Banach space with unit ball $\tilde{D}$ and $f: \tilde{D} \rightarrow D$ is a biholomorphic mapping. Then Proposition 1 also applies to $\tilde{D}$ and $\tilde{G} := \text{Aut}(\tilde{D})$. In particular, it follows from (iii) that $f(\tilde{G}(0)) = G(0)$, i.e. there exists a biholomorphic map $h: \tilde{D} \rightarrow D$ with $h(0) = 0$. But then h has to be linear. Q.E.D.

REMARK. For the proof of Proposition 1 we used only that $D \subset E$ is a bounded balanced domain except in the proof of Lemma 2 (where the application of Harris' result also requires that D be convex). Recently J. P. Vigué communicated to us without proof the following statement: For every bounded domain $D \subset E$ and every $a \in D$ the mapping $g \rightarrow E \times L(E)$ defined by $h(z)\partial/\partial z \mapsto (h(a), h'(a))$ has closed image. From this, Lemma 2 and, hence, Proposition 1 would follow for every bounded balanced domain $D \subset E$.

ADDED IN PROOF. The statement of Vigué appeared in J.-P. Vigué, C.R. Acad. Sci. Paris Sér. A **282** (1976), p. 113, Corollaire.

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