SOME APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM TO PLANAR RATIONAL APPROXIMATION

STEVEN MINSKER

ABSTRACT. The Stone-Weierstrass theorem is used to prove two general results about algebras of continuous functions, and each of these yields a necessary and sufficient condition for the planar function algebras $R(K)$ and $C(K)$ to be coincident.

We begin with a result which slightly extends the Stone-Weierstrass theorem.

THEOREM 1. (*Almost selfadjoint algebras are selfadjoint.*) Let X be a compact Hausdorff space, and let $f \in C(X)$. Suppose m and n are relatively prime positive integers. Let A be the closed subalgebra of $C(X)$ generated by f^m , $\bar{f}^n$, and the constant functions. Then $A = \{g \circ f: g \in C(f(X))\}$. In particular, if f separates the points of X , then $A = C(X)$.

PROOF. Let $K = f(X)$ and let B be the closed subalgebra of $C(K)$ generated by z^m , $\bar{z}^n$, and the constant functions. We claim $B = C(K)$. By the usual Stone-Weierstrass theorem, it suffices to show that $z \in B$ and $\bar{z} \in B$. Now $z^m \in B$ and $\bar{z}^n \in B$ imply $(z^m)^n (\bar{z}^n)^m \in B$, or $|z|^{2mn} \in B$. Since the function $h(u) = u^{1/2mn}$ is uniformly approximable by polynomials on the interval $[0, \|z\|_K^{2mn}]$ and B is closed, it follows that $|z| \in B$. Since $(m, n) = 1$, we can find integers a and b with $am + bn = 1$. Without loss of generality, we can assume that $a \geq 0$, $b \leq 0$. Then on $K - \{0\}$ we have

$$z = z^{am+bn} = (z^m)^a / (z^n)^{-b} = (z^m)^a (\bar{z}^n)^{-b} / |z|^{-2bn}$$

and

$$\bar{z} = z^n / \bar{z}^{n-1} = \bar{z}^n z^{n-1} / |z|^{2n-2}.$$

For each $\epsilon > 0$, let

$$h_\epsilon(u) = \begin{cases} 1/\epsilon^{-2bn} & \text{on } [0, \epsilon], \\ 1/u^{-2bn} & \text{on } [\epsilon, \|z\|_K], \end{cases}$$

Received by the editors September 25, 1975 and, in revised form, December 1, 1975.

AMS (MOS) subject classifications (1970). Primary 46J10; Secondary 46J30.

Key words and phrases. Stone-Weierstrass theorem, rational approximation, dirichlet algebra.

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and let $p_\epsilon(u)$ be a polynomial which uniformly approximates $h_\epsilon(u)$ within ϵ on the interval $[0, \|z\|_K]$. Consider the function $q_\epsilon(z) = (z^m)^a (\bar{z}^n)^{-b} p_\epsilon(|z|)$. Clearly $q_\epsilon \in B$. If $z_0 \in K$ and $|z_0| < \epsilon$, then

$$\begin{aligned} |q_\epsilon(z_0) - z_0| &\leq |q_\epsilon(z_0)| + |z_0| \\ &< \epsilon^{am-bn} (h_\epsilon(|z_0|) + \epsilon) + \epsilon = 2\epsilon + \epsilon^{am-bn+1}. \end{aligned}$$

If $z_0 \in K$ and $|z_0| \geq \epsilon$, then

$$\begin{aligned} |q_\epsilon(z_0) - z_0| &= |(z_0^m)^a (\bar{z}_0^n)^{-b}| \cdot |p_\epsilon(|z_0|) - h_\epsilon(|z_0|)| \\ &< \epsilon \|z\|_K^{am-bn}. \end{aligned}$$

Hence

$$\|q_\epsilon - z\|_K \rightarrow 0$$

as $\epsilon \rightarrow 0$, and so $z \in B$. In completely analogous fashion, if we let

$$\tilde{h}_\epsilon(u) = \begin{cases} 1/\epsilon^{2n-2} & \text{on } [0, \epsilon], \\ 1/u^{2n-2} & \text{on } [\epsilon, \|z\|_K], \end{cases}$$

let $\tilde{p}_\epsilon(u)$ be a polynomial which uniformly approximates $\tilde{h}_\epsilon(u)$ within ϵ on the interval $[0, \|z\|_K]$, and put $\tilde{q}_\epsilon(z) = \bar{z}^n z^{n-1} \tilde{p}_\epsilon(|z|)$, then $\tilde{q}_\epsilon \in B$ and $\|\tilde{q}_\epsilon - \bar{z}\|_K \rightarrow 0$ as $\epsilon \rightarrow 0$, so $\bar{z} \in B$. This establishes our claim that $B = C(K)$. Since any $g \in C(K)$ is thus uniformly approximable on K by polynomials in z^m and $\bar{z}^n$, it follows that $g \circ f$ is uniformly approximable on X by polynomials in f^m and $\bar{f}^n$. Hence $A \supset \{g \circ f: g \in C(f(X))\}$. The remaining assertions are obvious.

COROLLARY. *Let K be a compact subset of the complex plane. Then $R(K) = C(K)$ iff $\bar{z}^n \in R(K)$ for some $n \in \mathbb{N}$.*

PROOF. Immediate from the theorem.

REMARK. An analysis of the proof of Theorem 1 shows that the closed subalgebra of $C(X)$ generated by f^m and $\bar{f}^n$ contains both f and $\bar{f}$; that is, $\{f^m, \bar{f}^n\}$ and $\{f, \bar{f}\}$ generate the same closed subalgebra of $C(X)$. Consequently, the following extended Stone-Weierstrass theorem holds:

Let X be a compact Hausdorff space and let A be a subalgebra of $C(X)$ such that:

- (1) *if $x_0 \in X$, then there exists $f \in A$ with $f(x_0) \neq 0$,*
- (2) *A separates the points of X ,*
- (3) *if $f \in A$, then $\bar{f}^n \in A$ for some $n \in \mathbb{N}$ depending on f .*

Then A is dense in $C(X)$.

THEOREM 2. *Let X be a compact Hausdorff space, and let $f \in C(X)$. Let $\alpha > 0$. If A is the subalgebra of $C(X)$ generated by f , $|f|^\alpha$, and the constant functions, then $\text{Re } A$ (i.e., the real parts of functions in A) is dense in*

$\{g \circ f: g \in C_{\mathbf{R}}(f(X))\}$. In particular, if f separates the points of X , then A is a dirichlet algebra on X (i.e., $\text{Re } A$ is dense in $C_{\mathbf{R}}(X)$).

PROOF. Let $K = f(X)$, and let B be the closed subalgebra of $C(K)$ generated by z , $|z|^\alpha$, and the constant functions. We claim $\text{Re } B$ is dense in $C_{\mathbf{R}}(K)$. As in the proof of Theorem 1, $|z|^\alpha \in B$ implies $|z| \in B$. If j and k are nonnegative integers with $j \geq k$, then $\text{Re } (z^j \bar{z}^k) = \text{Re } (z^{j-k} |z|^{2k}) \in \text{Re } B$, and if $j < k$, then

$$\text{Re } (z^j \bar{z}^k) = \text{Re } (\overline{z^j \bar{z}^k}) = \text{Re } (z^k \bar{z}^j) = \text{Re } (z^{k-j} |z|^{2j}) \in \text{Re } B.$$

Analogously, $\text{Im } (z^j \bar{z}^k) = \text{Re } (-iz^j \bar{z}^k)$ implies that $\text{Im } (z^j \bar{z}^k) \in \text{Re } B$. By the Stone-Weierstrass theorem, the algebra generated by z , $\bar{z}$, and the constants is dense in $C(K)$. The density of $\text{Re } B$ in $C_{\mathbf{R}}(K)$ follows immediately, establishing our claim. Since any $g \in C_{\mathbf{R}}(K)$ is thus uniformly approximable on K by the real parts of polynomials in z and $|z|^\alpha$, $g \circ f$ is uniformly approximable on X by the real parts of polynomials in f and $|f|^\alpha$. This completes the proof.

COROLLARY. Let K be a compact subset of the complex plane. Then $R(K) = C(K)$ iff $|z|^\alpha \in R(K)$ for some $\alpha > 0$.

PROOF. By the preceding theorem, $|z|^\alpha \in R(K)$ implies that $R(K)$ is a dirichlet algebra on K . This in turn implies that every point of K is a peak point for $R(K)$. By a theorem of Bishop [1, Theorem 3.3.3], we conclude $R(K) = C(K)$.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PENNSYLVANIA 15260

Current address: Department of Mathematics, Rutgers University, Camden, New Jersey 08102