INTEGRAL REPRESENTATIONS OF CYCLIC GROUPS OF ORDER p^2

IRVING REINER¹

ABSTRACT. Let p be an odd prime, either regular or properly irregular, and let G be a cyclic group of order p^2 . The author determines a full set of inequivalent representations of G by matrices with rational integral entries. This information is used to calculate the ideal class number of the integral group ring ZG .

Let G be a cyclic group of order p^2 with generator x , where p is an odd prime. We shall determine the number $n(ZG)$ of nonisomorphic indecomposable ZG -lattices, for the case where p is a regular odd prime (that is, p does not divide the class number of the cyclotomic field $Q(\sqrt[p]{1})$). We shall also calculate $h(ZG)$, the number of isomorphism classes of ideals M in ZG for which $QM = QG$. Such problems were considered previously by various authors (see Heller-Reiner [2], for example), but no explicit formulas for $n(ZG)$ and $h(ZG)$ were obtained, because of difficulties about units. The methods of Galovich [1] enable one to overcome these difficulties for the case where p is a regular odd prime. We restrict ourselves to this case from now on, unless otherwise stated.

We may write the integral group ring ZG as $Z[x]/(x^{p^2} - 1)$. Letting $\phi_{p^m}(x)$ denote the cyclotomic polynomial of order p^m , we now set

$$R = Z[x]/(\phi_p(x)), \quad S = Z[x]/(\phi_{p^2}(x)), \quad T = Z[x]/(x^p - 1).$$

Then R is the ring of all algebraic integers in the cyclotomic field $Q(\sqrt[p]{1})$; let h_R denote its class number. Likewise, S is the ring of integers in $Q(\sqrt[p^2]{1})$; let its class number be h_S . Finally, T is the integral group ring of a group of order p .

We shall set $\lambda = 1 - x$, viewed either as an element of ZG , or else as an element of some quotient ring, such as R , S , R/pR , and so on. Let

$$\bar{Z} = Z/pZ, \quad \bar{R} = R/pR = R/\lambda^{p-1}R \cong \bar{Z}[\lambda]/(\lambda^{p-1}),$$

$$\bar{T} = T/pT \cong \bar{Z}[\lambda]/(\lambda^p).$$

The subscript p will indicate p -adic completion. In order to conform to the notation in [2], we now set

$$A = Z_p a \cong Z_p, \quad B = R_p b \cong R_p, \quad C = S_p c \cong S_p, \quad E = T_p e \cong T_p,$$

where A , B , C and E are viewed as $Z_p G$ -lattices. For each ZG -lattice N , we have

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$$\text{Ext}_{ZG}^1(S, N) \cong \text{Ext}_{Z_p G}^1(S_p, N_p) \cong \bar{N} = N/pN.$$

As shown in [2], a full list of nonisomorphic indecomposable $Z_p G$ -lattices is as follows:

- (1) A, B, C, E .
- (2) Extensions of C by $A \oplus E$, corresponding to extension classes $\bar{a} + \lambda^r \bar{e}$ in $\bar{A} \oplus \bar{E}$, $1 \leq r \leq p-2$.
- (3) Extensions of C by E , corresponding to extension classes $\lambda^r \bar{e}$, $0 \leq r \leq p-1$.
- (4) Extensions of C by $A \oplus B$; classes $\bar{a} + \lambda^r \bar{b}$, $0 \leq r \leq p-2$.
- (5) The extension of C by A , with extension class $\bar{a}$.
- (6) Extensions of C by B , corresponding to extension classes $\lambda^r \bar{b}$, $0 \leq r \leq p-2$.

For each indecomposable ZG -lattice X , we know from [2] that its completion X_p is also indecomposable, and hence occurs in the above list. We shall say that X is of "type k " if X_p is listed in the k th grouping. Corresponding to the $Z_p G$ -lattice A , the only possible X is the G -trivial module Z . Next, the h_R representatives of the ideal classes of R give (up to isomorphism) all indecomposable ZG -lattices X for which $X_p \cong B$. Likewise, there are h_S ideals of S which correspond to C . Finally, there are precisely h_R possible X 's corresponding to T , namely, the projective ideals of the integral group ring T . (This last assertion follows readily from the discussion below.)

Consider next an X of type (6), so X_p is an extension of C by B . Then X must be isomorphic to an extension of $\mathfrak{c}$ by $\mathfrak{b}$, where $\mathfrak{c}$ is an ideal of S , and $\mathfrak{b}$ is an ideal of R . Now

$$\text{Ext}_{ZG}^1(\mathfrak{c}, \mathfrak{b}) \cong \text{Ext}_{ZG}^1(S, R) \cong \bar{R}.$$

Furthermore,

$$\text{Aut } \mathfrak{c} \cong \text{Aut } S \cong u(S), \quad \text{Aut } \mathfrak{b} \cong \text{Aut } R \cong u(R),$$

where Aut denotes the group of ZG -automorphisms, and $u(\)$ denotes the group of units. However, the isomorphism class of X is completely determined by the orbit of its extension class under the action of $\text{Aut } \mathfrak{c}$ and $\text{Aut } \mathfrak{b}$. It follows at once that in enumerating such orbits, we need only consider the case where X is an extension of S by R . The total number of X 's will then be $h_R h_S$ times the number of extensions of S by R .

So now let X be an extension of S by R , and let $\xi \in \bar{R}$ be its class in $\text{Ext}^1(S, R)$. We may write $\xi = \lambda^r v$, where $0 \leq r \leq p-2$ and where v is a unit in the ring $R' = \bar{Z}[\lambda]/(\lambda^{p-1-r})$. We must calculate the number of orbits of $u(R')$ under the action of $u(R)$ and $u(S)$. For each $w \in u(S)$, its norm $N(w)$ from S to R is a unit of R . It is easily verified that w acts on $u(R')$ in the same way that $N(w)$ acts. Hence it suffices for us to count the orbits of $u(R')$ under the action of $u(R)$.

Now $u(R')$ is the direct product of a cyclic group D of order $p-1$, and an elementary abelian p -group generated by the units

$$1 + \lambda, 1 + \lambda^2, 1 + \lambda^3, \dots, 1 + \lambda^{p-2-r}.$$

But for p regular, Galovich [1] showed that the image of $u(R)$ in $u(R')$ is the product of D and a subgroup generated by units of the form

$$(6a) \quad 1 + \lambda, 1 + \lambda^2 + \alpha_2 \lambda^3, 1 + \lambda^4 + \alpha_4 \lambda^5, \\ 1 + \lambda^6 + \alpha_6 \lambda^7, \dots, 1 + \lambda^k + \alpha_k \lambda^{k+1},$$

where each $\alpha_i \in R'$, and where k is the largest even integer $\leq p - 2 - r$. The "missing" exponents are 3, 5, 7, $\dots$, $p - 2 - r$, the last term occurring only for even r . The number of such missing exponents is $[(p - 3 - r)/2]$, where this symbol² is interpreted as 0 whenever $p - 3 - r < 0$. Consequently, the cokernel of the map $u(R) \rightarrow u(R')$ is an elementary p -group on $[(p - 3 - r)/2]$ generators. Hence the total number of indecomposable ZG -lattices of type (6) is given by

$$(7) \quad h_R h_S \alpha, \quad \text{where } \alpha = \sum_{r=0}^{p-2} p^{[(p-3-r)/2]}.$$

A similar argument shows that the number of lattices of type (4) is also given by formula (7). In this situation, we must count the orbits of $\bar{Z} \oplus \bar{R}$ under the action of $u(R)$ and $u(S)$. This action is given by

$$w(\bar{z} + \bar{r})u = N(w)\bar{z} + N(w)\bar{r}u$$

for $w \in u(S)$, $\bar{z} + \bar{r} \in \bar{Z} \oplus \bar{R}$, $u \in u(R)$. We may choose w so that $N(w)\bar{z} = \bar{1}$, and then proceed with the remaining term $N(w)\bar{r}u$ as in the preceding paragraph. This gives formula (7) just as before.

It is easily seen that there are h_S indecomposable ZG -lattices of type (5), and it remains for us to consider lattices of types (2) and (3). Suppose that X is of type (3), and is an extension of S by T . We have seen that

$$\text{Ext}^1(S, T) \cong \bar{T} \cong \bar{Z}[\lambda]/(\lambda^p),$$

and the extension class of X is of the form $\lambda^r v$ for some r , $0 \leq r \leq p - 1$, and some $v \in u(R')$ with $R' = \bar{Z}[\lambda]/(\lambda^{p-r})$. We wish to count the number of orbits of $u(R')$ under the action of $u(S)$ and $u(T)$.

If $r \geq 1$ then R' is a quotient of R , and both $u(S)$ and $u(T)$ act on R' via $u(R)$. Hence by the preceding discussion, the number of orbits equals $p^{[(p-2-r)/2]}$. Consider now the case $r = 0$, in which $R' = \bar{Z}[\lambda]/(\lambda^p) = \bar{T}$. The group $u(\bar{T})$ is the direct product of a cyclic group D of order $p - 1$, and an elementary p -group with generators

$$1 + \lambda, 1 + \lambda^2, 1 + \lambda^3, \dots, 1 + \lambda^{p-1}.$$

On the other hand, we note that

$$\bar{T} = Z[x]/(\phi_{p^2}(x), x^p - 1) \cong S/(x^p - 1) = S/(\lambda^p).$$

By [1], the image of $u(S)$ in $u(\bar{T})$ is the product of D and the subgroup generated by units of the form

$$1 + \lambda, 1 + \lambda^2 + \alpha_2 \lambda^3, 1 + \lambda^4 + \alpha_4 \lambda^5, \dots, 1 + \lambda^{p-1},$$

where each $\alpha_i \in \bar{T}$. The "missing" exponents are 3, 5, $\dots$, $p - 2$; there are $(p - 3)/2$ such exponents. Furthermore, none of them can be recaptured from the action of $u(T)$ on $u(\bar{T})$, since each unit of T maps onto a unit of R under the surjection $T \rightarrow R$. Thus the number of orbits for the case $r = 0$ is

²For $y > 0$, the symbol $[y]$ denotes the greatest integer $< y$.

precisely $p^{(p-3)/2}$. It follows at once that the total number of indecomposable ZG -lattices of type (3) is given by

$$(8) \quad h_R h_S \beta, \quad \text{where } \beta = \sum_{r=0}^{p-1} p^{[(p-2-r)/2]}.$$

A similar argument shows that the number of lattices of type (2) is given by $(\alpha - 1)h_R h_S$, since in (2) the parameter r ranges from 1 to $p - 2$. We may remark that

$$\beta = \alpha + p^{(p-3)/2},$$

where α is defined in (7).

Combining all of these results, we have:

THEOREM. *For p a regular odd prime, the number $n(ZG)$ of isomorphism classes of indecomposable ZG -lattices is given by*

$$n(ZG) = 1 + 2h_R + 2h_S + (3\alpha - 1)h_R h_S + \beta h_R h_S,$$

where α, β are defined in (7) and (8).

We are now in a position to compute $h(ZG)$, the number of isomorphism classes of ideals M in ZG for which $QM = QG$. In the first column below, we list all M 's (apart from using ideals in R and S , rather than the rings R and S themselves). In the second column, we list the number of M 's of each kind, omitting the factor $h_R h_S$ arising from the ideal classes.

Classes of ideals in ZG	Number of classes
$Z \oplus R \oplus S$	1
$Z \oplus \text{type 6}$	α
$R \oplus \text{type 5}$	1
$S \oplus E$	1
type 3	β
type 4	α

(Note that type 2 does not give ideals in ZG .) We thus obtain:

COROLLARY. *The number of ideal classes of ZG is*

$$h(ZG) = (3 + 2\alpha + \beta)h_R h_S$$

with α, β as in (7) and (8).

When the prime p is irregular, the expressions in the Theorem and its Corollary give *lower* bounds for $n(ZG)$ and $h(ZG)$, respectively. This follows from our earlier discussion and Galovich's results about units for the case of irregular primes.

The prime p is called *properly irregular* if p divides h_R , but does not divide the class number of the maximal real subfield of $Q(\sqrt[p]{1})$. As pointed out by Galovich, his methods can be applied to the case of properly irregular primes to obtain analogues of the preceding Theorem and its Corollary. For $k \leq (p-3)/2$, let $\delta(k)$ be the number of Bernoulli numbers among $B_1, B_2, \dots, B_k$ whose numerators are multiples of p . (If p is a regular prime, each

$\delta(k) = 0$.) Assuming now that p is properly irregular, let us consider the orbits of $u(R')$ under the action of $u(R)$, where $R' = \overline{Z}[\lambda]/(\lambda^{p-1}-r)$, with $0 \leq r \leq p-2$. In determining the image of $u(R)$ in $u(R')$, we must omit from the list (6a) all those terms $1 + \lambda^{2i} + \alpha_i \lambda^{2i+1}$ for which p divides the numerator of B_i . It follows at once that the number of indecomposable ZG-lattices of type (6) is given by

$$h_R h_S \alpha', \quad \text{where } \alpha' = \sum_{r=0}^{p-2} p^{[(p-3-r)/2] + \delta[(p-2-r)/2]}.$$

This also gives the number of indecomposables of type (4).

On the other hand, the image of $u(S)$ in $u(\overline{T})$ still contains $1 + \lambda^{p-1}$, even when p is properly irregular. It follows easily that the number of indecomposable lattices of type (3) is given by

$$h_R h_S \beta', \quad \text{where } \beta' = \alpha' + p^{(p-3)/2 + \delta((p-3)/2)}.$$

Finally, the number of indecomposables of type (2) is given by $h_R h_S (\alpha' - 1)$. Thus, when p is properly irregular, the analogues of the Theorem and its Corollary are valid, provided that we use α' in place of α , and β' in place of β .

As a concluding remark, it should be mentioned that in all cases known so far, every odd prime is either regular or properly irregular.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801