COUNTEREXAMPLES ON THE FUSION OF INVOLUTIONS IN FINITE GROUPS

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ABSTRACT. Some examples are given of simple groups G containing a conjugacy class of involutions $\mathcal{C}$ such that if S is a Sylow 2-subgroup of G , then there exist subgroups $A \subseteq B \subseteq S$ with $|A : B| = 2$ and $\mathcal{C} \cap S \subseteq B - A$.

Three conjectures arise naturally in the study of the fusion of involutions in finite groups.

Conjecture 1. Let G be a finite group. Let $\mathcal{C}$ be a union of conjugacy classes of involutions and let $S \in \text{Syl}_2(G)$. For $H \subseteq G$, define $K_H = \text{gp} \langle ab | a, b \in \mathcal{C} \cap H \rangle$. Then $\mathcal{C} \cap K_S = \emptyset$ implies $\mathcal{C} \cap K_G = \emptyset$.

Conjecture 2. Let G be a finite group and let $\mathcal{C}$ and $\mathcal{D}$ each be unions of conjugacy classes of involutions. Let $S \in \text{Syl}_2(G)$. Then $\mathcal{C} \cap \langle \mathcal{D} \cap S \rangle = \emptyset$ implies $\mathcal{C} \cap \langle \mathcal{D} \rangle = \emptyset$.

Conjecture 3. Let G be a finite group and let $\mathcal{C}$ and $\mathcal{D}$ each be unions of conjugacy classes of involutions. Let $S \in \text{Syl}_2(G)$. For $H \subseteq G$, let $L_H = \langle a, bc | a \in \mathcal{D} \cap H, b, c \in \mathcal{C} \cap H \rangle$. Then $L_S \cap \mathcal{C} = \emptyset$ implies $\langle \mathcal{D} \rangle \cap \mathcal{C} = \emptyset$.

In the special case when $K_S = 1$, Conjecture 1 is equivalent to Glauberman's Z^* theorem; in the case when $|S : K_S| = 2$, Conjecture 1 is equivalent to the Thompson transfer lemma. Conjecture 1 is an attempt to generalize these results.

Conjecture 3 is not only an immediate consequence of Conjecture 2; it is also a special case of Conjecture 1. Namely, assuming Conjecture 1 and the hypotheses of Conjecture 3, let $G_0 = \langle z \rangle \times G$, where $z^2 = 1$, let $\mathcal{C}_0 = \mathcal{C} \cup z\mathcal{D}$ and apply Conjecture 1 to G_0 and $\mathcal{C}_0$ to derive the conclusion of Conjecture 3. The purpose of this note is to point out an infinite series of counterexamples to Conjecture 3 and hence to Conjectures 1 and 2.

The groups to be discussed are the simple groups $\text{PSL}_8(q)$ where q is a prime power with $q \equiv 1 \pmod{16}$. By Dirichlet's theorem, there are infinitely many suitable q , of which the smallest is $q = 17$. The example is based upon consideration of the natural representation of $\text{SL}_8(q)$ on an 8-dimensional vector space over $\text{GF}(q)$. Throughout the discussion, therefore, let G

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$= \text{SL}_8(q)$, $\mathbf{Z} = \mathbf{Z}(\text{SL}_8(q))$, $\bar{G} = \text{PSL}_8(q) = G/\mathbf{Z}$ with the usual bar convention for subgroups. Let V be an 8-dimensional vector space over $\text{GF}(q)$ which faithfully represents G and fix an $S \in \text{Syl}_2(G)$.

The first step is to show S is monomial on V . Let $F_1, F_2, \dots, F_8$ be a collection of 1-dimensional subspaces of V which span V and let G_0 be the subgroup of all unimodular transformations which permute the subspaces $\{F_i | 1 \leq i \leq 8\}$ and such that the induced permutations preserve the systems of imprimitivity $\{\{F_1, \dots, F_4\}, \{F_5, \dots, F_8\}\}$ and $\{\{F_1, F_2\}, \{F_3, F_4\}, \{F_5, F_6\}, \{F_7, F_8\}\}$. If $x \in G_0$, let x^π be the corresponding permutation of the $\{F_i | 1 \leq i \leq 8\}$ and let D_0 be the diagonal subgroup of group elements fixing all F_i . Then G_0/D_0 is isomorphic to $Z_2 \sim (Z_2 \times Z_2)$, a Sylow 2-subgroup of A_8 . Fix an $S_0 \in \text{Syl}_2(G_0)$ and let $D = S_0 \cap D_0$, so that $S_0/D \cong Z_2 \sim (Z_2 \times Z_2)$. Since S_0 is transitive on $\{F_i | 1 \leq i \leq 8\}$, $\mathbf{C}_{\text{GL}_8(q)}(S_0) = \mathbf{Z}(\text{GL}_8(q))$ so that the representation of S_0 on V is absolutely irreducible. We may choose $S \in \text{Syl}_2(G)$ with $S \supseteq S_0$ and then the representation of S on V is absolutely irreducible. Since $(q, 2) = 1$ and $\text{GF}(q)$ contain a primitive 4-root of 1, the representation of S on V is monomial and this forces $S = S_0$ [1, (14.4)]. This equality could also be derived purely from group order considerations.

Let λ be a primitive 16-root of 1 in $\text{GF}(q)$, fix nonzero basis elements $f_i \in F_i$ for $1 \leq i \leq 8$ and in this basis define elements of G ;

$$c = \text{diag}(\lambda, \lambda, \lambda, \lambda, \lambda, \lambda, \lambda, -\lambda), \quad d = \text{diag}(1, 1, 1, 1, 1, 1, -1, -1).$$

Let $\mathcal{C} = \text{ccl}(c)$, $\mathcal{D} = \text{ccl}(d)$. Let T be the stabilizer in S of the sets $\{F_1, \dots, F_4\}$ and $\{F_5, \dots, F_8\}$.

The second step is to observe that if $h \in \mathcal{D} \cap S$ then $h \in T$. For otherwise, h^π consists of four transpositions. h fixes four corresponding 2-dimensional subspaces; but then, since h has 6 eigenvalues 1, h is the identity on one space $\langle F_u, F_v \rangle$, $1 \leq u < v \leq 8$, contradicting the fact that h^π is a transposition on $\{F_u, F_v\}$. Similarly, if $g \in \mathcal{C} \cap S$, then $g \in T$.

Now define a homomorphism $\delta: T\mathbf{Z} \rightarrow Z_{2^a}$ by $\delta(x) = \det x|_{\langle F_1, \dots, F_4 \rangle}$. If $z \in \mathbf{Z}$ then $\delta(z) = \pm 1$. Also $\delta(h) = \pm 1$ for $h \in \mathcal{D} \cap S (\subseteq T)$. However, if $g \in \mathcal{C} \cap S$ then $\delta(g) = \pm \lambda^4$. Let $L = \delta^{-1}(\pm 1)$. Then $\mathcal{C} \cap L = \emptyset$, but if $g, h \in \mathcal{C} \cap S$ then $\delta(gh) = \pm \lambda^8 = \pm 1$ so $gh \in L$. In $\bar{G}$, $\mathcal{C}$ and $\mathcal{D}$ become conjugacy classes of involutions and if L_S is defined as in Conjecture 3 then $\bar{L} \supseteq L_S$. $\mathcal{C} \cap \bar{L} = \emptyset$ so $\mathcal{C} \cap L_S = \emptyset$. However $\langle \mathcal{D} \rangle = \bar{G}$, by the simplicity of $\text{PSL}_8(q)$. This example disproves all the conjectures. In particular, the class $\mathcal{C}$ alone already suffices to disprove Conjecture 1.

REFERENCES

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