CONTINUOUS DEPENDENCE OF LEAST SQUARES SOLUTIONS OF LINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. Let u_λ be the unique least squares solution of minimal norm of the linear boundary value problem $Lu - \lambda u = f$, where L is a selfadjoint differential operator in $L^2[a, b]$. Working in the Sobolev space $H^n[a, b]$, the alternative method is used to examine the continuous dependence of the u_λ on the parameter λ as $\lambda \rightarrow \lambda_0$, and the convergent and divergent cases are both characterized.

1. Introduction. Let S be the real Hilbert space $L^2[a, b]$ with the standard inner product (u, v) and norm $\|u\|$, and let $H^n[a, b]$ be the subspace consisting of all functions $u(t)$ in $C^{n-1}[a, b]$ with $u^{(n-1)}$ absolutely continuous on $[a, b]$ and $u^{(n)}$ in S . This subspace becomes a Banach space under the norm

$$\|u\|_n = \sum_{i=0}^{n-1} \max_{a \leq t \leq b} |u^{(i)}(t)| + \|u^{(n)}\|, \quad u \in H^n[a, b].$$

We refer to the topology induced by this norm as the H^n -topology and denote convergence by $u_i \rightarrow^{H^n} u$.

Let

$$\tau = \sum_{i=0}^n a_i(t) \left(\frac{d}{dt} \right)^i$$

be an n th order formal differential operator with coefficients $a_i(t)$ in $C^\infty[a, b]$ and $a_n(t) \neq 0$ on $[a, b]$, let

$$B_i(u) = \sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij} u^{(j)}(b), \quad i = 1, \dots, n,$$

be a set of linearly independent boundary values, and let L be the differential operator in S defined by

$$\mathcal{D}(L) = \{u \in H^n[a, b] | B_i(u) = 0, i = 1, \dots, n\}, \quad Lu = \tau u.$$

We assume that L is selfadjoint, and let ϕ_i , $i = 1, 2, \dots$, be an orthonormal basis for S made up of eigenfunctions of L with corresponding eigenvalues λ_i , $i = 1, 2, \dots$, so that $L\phi_i = \lambda_i\phi_i$ and $|\lambda_i| \rightarrow \infty$.

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For a fixed function $f \in S$ and a real number λ , we consider the linear boundary value problem

$$(1) \quad Lu - \lambda u = f.$$

If we let $L_\lambda = L - \lambda I$ and let P_λ be the orthogonal projection of S onto the null space $\mathfrak{N}(L_\lambda)$, then $I - P_\lambda$ is the orthogonal projection of S onto the range $\mathfrak{R}(L_\lambda) = \mathfrak{N}(L_\lambda)^\perp$, and the following well-known theorem characterizes the solvability of equation (1).

THEOREM 1. *The boundary value problem (1) is solvable iff $P_\lambda f = 0$. Thus, if $\lambda \neq \lambda_i$ for $i = 1, 2, \dots$, then (1) is uniquely solvable, while if $\lambda = \lambda_i$ for some $i \geq 1$, then (1) either has no solutions or it has infinitely many solutions.*

In this paper we are interested in studying least squares solutions of equation (1), and in particular, in determining their dependence on the parameter λ . Let

$$H_\lambda = [L_\lambda | \mathfrak{D}(L_\lambda) \cap \mathfrak{N}(L_\lambda)^\perp]^{-1},$$

and let $K_\lambda = H_\lambda(I - P_\lambda)$. The linear operator $K_\lambda: S \rightarrow S$ is the generalized inverse of L_λ , and the function $u_\lambda = K_\lambda f$ is the unique least squares solution of equation (1) of minimal L^2 -norm. In case equation (1) is solvable, u_λ is an actual solution. These ideas have been discussed in [3].

For a fixed real number λ_0 we ask if $u_\lambda \xrightarrow{H^n} u_{\lambda_0}$ as $\lambda \rightarrow \lambda_0$? We certainly would expect this to be true if λ_0 is not an eigenvalue of L , while the situation when λ_0 is an eigenvalue seems more obscure. In the next section we utilize the alternative method (see [1] or [2]) to characterize when the u_λ are H^n -convergent to u_{λ_0} , at the same time obtaining error estimates, and for the divergent case we describe the manner in which the divergence occurs.

2. Continuous dependence. Fixing the real number λ_0 , we set

$$\delta = \inf_{\lambda_i \neq \lambda_0} |\lambda_i - \lambda_0| > 0.$$

For $i = 1, 2, \dots$ we observe that $L_{\lambda_0}\phi_i = 0$ and $K_{\lambda_0}\phi_i = 0$ for $\lambda_i = \lambda_0$, and $L_{\lambda_0}\phi_i = (\lambda_i - \lambda_0)\phi_i$ and $K_{\lambda_0}\phi_i = (\lambda_i - \lambda_0)^{-1}\phi_i$ for $\lambda_i \neq \lambda_0$, and hence, K_{λ_0} is a diagonal operator on S with $\|K_{\lambda_0}\| = 1/\delta$. Also, since K_{λ_0} is continuous from S under the L^2 -topology to $H^n[a, b]$ under the H^n -topology (see [3, p. 180]), we can choose a constant $\gamma > 0$ depending on λ_0 such that

$$(2) \quad \|K_{\lambda_0}u\|_n \leq \gamma\|u\| \quad \text{for all } u \in S.$$

For each real number λ let $u_\lambda = v_\lambda + w_\lambda$ where $v_\lambda \in \mathfrak{N}(L_{\lambda_0})$ and $w_\lambda \in \mathfrak{N}(L_{\lambda_0})^\perp = \mathfrak{R}(L_{\lambda_0})$. We know that for λ satisfying $0 < |\lambda - \lambda_0| < \delta$, the function u_λ is the unique solution of equation (1), and hence,

$$(3) \quad L_{\lambda_0}u_\lambda + (\lambda_0 - \lambda)u_\lambda - f = 0.$$

By the alternative method it follows that for $0 < |\lambda - \lambda_0| < \delta$ the functions u_λ and v_λ satisfy the equivalent pair of equations

$$(4) \quad u_\lambda + K_{\lambda_0}[(\lambda_0 - \lambda)u_\lambda - f] = v_\lambda,$$

$$(5) \quad P_{\lambda_0}[(\lambda_0 - \lambda)u_\lambda - f] = 0.$$

THEOREM 2. *The boundary value problem (1) is solvable for $\lambda = \lambda_0$ iff the norms $\|u_\lambda\|$ remain bounded as $\lambda \rightarrow \lambda_0$, in which case $v_\lambda = 0$ and $u_\lambda = w_\lambda$ for $|\lambda - \lambda_0| < \delta$, and $u_\lambda \rightarrow^{H^n} u_{\lambda_0}$ as $\lambda \rightarrow \lambda_0$ with the error estimate*

$$(6) \quad |u_\lambda - u_{\lambda_0}|_n \leq (2\gamma/\delta) \|f\| |\lambda - \lambda_0| \quad \text{for all } |\lambda - \lambda_0| < \delta/2.$$

PROOF. First, assume (1) is solvable for $\lambda = \lambda_0$, so $P_{\lambda_0}f = 0$ by Theorem 1. From equation (5) we obtain

$$0 = (\lambda_0 - \lambda)P_{\lambda_0}u_\lambda = (\lambda_0 - \lambda)v_\lambda,$$

and hence, $v_\lambda = 0$ for $0 < |\lambda - \lambda_0| < \delta$. Since $u_{\lambda_0} = K_{\lambda_0}f$, we get immediately that $v_{\lambda_0} = P_{\lambda_0}u_{\lambda_0} = 0$. If $|\lambda - \lambda_0| < \delta/2$, then from (4) we have

$$\|u_\lambda\| \leq (1/\delta)(\delta/2)\|u_\lambda\| + (1/\delta)\|f\|,$$

or

$$(7) \quad \|u_\lambda\| \leq (2/\delta)\|f\| \quad \text{for all } |\lambda - \lambda_0| < \delta/2.$$

Therefore, the norms $\|u_\lambda\|$ remain bounded as $\lambda \rightarrow \lambda_0$.

Second, assume the norms $\|u_\lambda\|$ are bounded as $\lambda \rightarrow \lambda_0$. From (5) we have

$$\|P_{\lambda_0}f\| \leq |\lambda - \lambda_0| \|u_\lambda\|,$$

and letting $\lambda \rightarrow \lambda_0$ we conclude that $P_{\lambda_0}f = 0$. By Theorem 1, equation (1) is solvable for $\lambda = \lambda_0$. To obtain the error estimate we simply note that (4) can be rewritten as

$$u_\lambda - u_{\lambda_0} = (\lambda - \lambda_0)K_{\lambda_0}u_\lambda,$$

and now apply (2) and (7). This completes the proof.

COROLLARY. *If λ_0 is not an eigenvalue of L , then for all λ satisfying $|\lambda - \lambda_0| < \delta$ the boundary value problem (1) has u_λ as its unique solution. Moreover, $u_\lambda \rightarrow^{H^n} u_{\lambda_0}$ as $\lambda \rightarrow \lambda_0$, with the rate of convergence determined by equation (6).*

THEOREM 3. *If the boundary value problem (1) is not solvable for $\lambda = \lambda_0$, then*

$$(8) \quad \|v_\lambda\| = |\lambda - \lambda_0|^{-1} \|P_{\lambda_0}f\| \rightarrow \infty \quad \text{as } \lambda \rightarrow \lambda_0,$$

$\|u_\lambda\| \rightarrow \infty$ as $\lambda \rightarrow \lambda_0$, and $w_\lambda \rightarrow^{H^n} u_{\lambda_0} = w_{\lambda_0}$ as $\lambda \rightarrow \lambda_0$ with the error estimate

$$(9) \quad |w_\lambda - u_{\lambda_0}|_n \leq (2\gamma/\delta) \|f\| |\lambda - \lambda_0| \quad \text{for all } |\lambda - \lambda_0| < \delta/2.$$

PROOF. We obtain (8) from (5) and Theorem 1, and since $\|v_\lambda\| \leq \|u_\lambda\|$, it follows that $\|u_\lambda\| \rightarrow \infty$ as $\lambda \rightarrow \lambda_0$. Now $K_{\lambda_0}u_\lambda = K_{\lambda_0}w_\lambda$, and hence, we can rewrite (4) as

$$(10) \quad w_\lambda + K_{\lambda_0}[(\lambda_0 - \lambda)w_\lambda - f] = 0$$

for $|\lambda - \lambda_0| < \delta$. Proceeding as in Theorem 2 we get the bound

$$(11) \quad \|w_\lambda\| \leq (2/\delta)\|f\| \quad \text{for all } |\lambda - \lambda_0| < \delta/2,$$

and a combination of (10) and (11) yields the estimate (9).

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