ON THE OPERATOR EQUATION $TX - XV = A$

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ABSTRACT. A characterization of the operators A for which the equation $TX - XV = A$ is solvable is given, where T is a fixed right invertible operator and V is a fixed unilateral shift.

The aim of this note is to give a characterization of the solutions of the equation

$$TX - XV = A,$$

where T , V , A are given operators acting in a Hilbert space, and V is a unilateral shift. As a by-product we also give a sufficient condition for A to be expressed in the form

$$A = VX - XV = V^*Y - YV^* = V^*Z - ZV$$

with V a unilateral shift.

The problems studied throughout the paper originate from a question of C. Foiaş.

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Let H be a complex Hilbert space and denote by $L(H)$ the algebra of all bounded linear operators acting in H . For any $T, S, A \in \mathcal{L}(H)$ put

$$d_n(A; T, S) = \sum_{j+k=n} T^j A S^k, \quad n \geq 0.$$

It is easy to see what we have

$$T d_n(A; T, S) - d_n(A; T, S) S = d_n(TA - AS; T, S) = T^{n+1} A - A S^{n+1},$$

$$d_{n+1}(A; T, S) = T d_n(A; T, S) + A S^{n+1} = d_n(A; T, S) S + T^{n+1} A.$$

1. **PROPOSITION.** Let $B \in \mathcal{L}(H)$ and let V be a unilateral shift such that $TB - BV = 0$, $B(I - VV^*) = 0$. Then we have $B = 0$.

PROOF. By induction we derive $BV^n = T^n B$, so $BV^n(I - VV^*) = 0$, $n \geq 0$, and this implies $B = 0$.

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2. LEMMA. Let $T, A \in \mathcal{L}(H)$ be given and let V be a unilateral shift. Then the following conditions are equivalent:

- (i) there exists $B \in \mathcal{L}(H)$ such that $TB - BV = A$, $B(I - VV^*) = 0$,
- (ii) there exists $c > 0$, such that for any sequence $\{x_k\}_{k=0}^\infty \subset \text{Ker } V^*$ and for any $n \geq 0$, we have

$$\left\| \sum_{k=0}^n d_k(A; T, V)x_k \right\| \leq c \left(\sum_{k=0}^n \|x_k\|^2 \right)^{1/2},$$

- (iii) the series $\sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k}$ is strongly convergent.

PROOF. (i) $\Rightarrow$ (ii). Let $\{x_k\}_{k=0}^\infty \subset \text{Ker } V^*$ and take $c = \|B\|$. We have

$$\begin{aligned} \left\| \sum_{k=0}^n d_k(A; T, V)x_k \right\|^2 &= \left\| \sum_{k=0}^n d_k(TB - BV; T, V)x_k \right\|^2 \\ &= \left\| \sum_{k=0}^n (T^{k+1}B - BV^{k+1})x_k \right\|^2 = \left\| -B \sum_{k=0}^n V^{k+1}x_k \right\|^2 \\ &\leq \|B\|^2 \left\| \sum_{k=0}^n V^{k+1}x_k \right\|^2 = c^2 \sum_{k=0}^n \|x_k\|^2. \end{aligned}$$

(ii) $\Rightarrow$ (iii). For any $x \in H$ we have

$$\begin{aligned} \left\| \sum_{k=m}^{m+n} d_k(A; T, V)(I - VV^*)V^{*k}x \right\|^2 &\leq c^2 \sum_{k=m}^{m+n} \|(I - VV^*)V^{*k}x\|^2 \\ &= c^2 \sum_{k=m}^{m+n} \|V^k(I - VV^*)V^{*k}x\|^2 = c^2 \|V^{m+n}V^{*m+n}x - V^{m+1}V^{*m+1}x\|^2 \end{aligned}$$

and because $\{V^{*k}\}_{k=0}^\infty$ tends strongly to 0, the strong convergence of the series $\sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k}$, follows.

(iii) $\Rightarrow$ (i). If we take $B = -(\sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k})V^*$, we have

$$\begin{aligned} TB - BV &= \sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k} \\ &\quad - T \sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k+1} \\ &= \sum_{k=0}^\infty d_k(A; T, V)(I - VV^*)V^{*k} + A \sum_{k=0}^\infty V^{k+1}(I - VV^*)V^{*k+1} \\ &\quad - \sum_{k=0}^\infty d_{k+1}(A; T, V)(I - VV^*)V^{*k+1} \\ &= d_0(A; T, V)(I - VV^*) + A \sum_{k=0}^\infty V^{k+1}(I - VV^*)V^{*k+1} = A. \end{aligned}$$

3. THEOREM. Let $T, A \in \mathcal{L}(H)$ be given and let V be a unilateral shift. Then the equations

$$TB - BV = A, \quad B(1 - VV^*) = 0$$

have a simultaneous solution B if and only if there exists $c > 0$ as in Lemma 2(ii); the solution B is unique.

PROOF. Existence follows from Lemma 2, and uniqueness from Proposition 1.

4. THEOREM. Let $T, A' \in \mathcal{L}(H)$ and let V be a unilateral shift. Suppose that T is right invertible. Then the equation $TB' - B'V = A'$ has a solution B' if and only if A' is of the form $A' = A + A_0$, with A fulfilling condition (ii) of Lemma 2 and $A_0(1 - VV^*) = A_0$.

PROOF. Suppose that there exists $B' \in \mathcal{L}(H)$ such that $TB' - B'V = A'$. Taking $A = TB'VV^* - B'V$, $A_0 = TB'(I - VV^*)$, the properties of A and A_0 stated in our theorem can be easily checked (see also Theorem 3). Conversely, if $A' = A + A_0$ and A, A_0 have the properties stated in our theorem, then taking B_0 such that $TB_0 = A_0$, $B_0V = 0$ (this is possible because T is right invertible) and taking B given by Theorem 3, the operator $B' = B + B_0$ will be a solution of the equation $T'B' - B'V = A'$. Recall now that the essential reducing spectrum of $A \in \mathcal{L}(H)$ is the set of all complex numbers λ for which there exists an orthonormal sequence $\{e_n\} \subset H$ such that (see [2])

$$\overline{\lim_{n \rightarrow \infty}} \|(A - \lambda)e_n\| = \overline{\lim_{n \rightarrow \infty}} \|(A - \lambda)^*e_n\| = 0.$$

5. THEOREM. Let $A \in \mathcal{L}(H)$ have 0 in its essentially reducing spectrum. Then there exists a unilateral shift V such that the equations

$$VX - XV = V^*Y - YV^* = V^*Z - ZV = A$$

are solvable.

PROOF. Let V be a unilateral shift and denote by P_n the orthogonal projection of H onto V^nH . Because 0 belongs to the essential reducing spectrum of A , we may choose V such that

$$\max\{\|P_n A\|, \|A P_n\|\} \leq \|A\|/2^n.$$

Let us put $Q_n = P_n - P_{n+1}$. Then we have $\sum_{j+k=0}^{\infty} \|Q_j A Q_k\| < \infty$ and the existence of X and Y follows from [1, Lemmas 1.1 and 1.3]. On the other hand, we have

$$\|d_n(A; V^*, V)\| \leq \sum_{j+k=0}^n \|P_j A P_k\| \leq (n+1)\|A\|/2^n;$$

thus if $\{x_k\}_{k=0}^{\infty} \subset \text{Ker } V^*$ is given, we obtain

$$\begin{aligned} \left\| \sum_{k=0}^n d_k(A; V^*, V)x_k \right\| &\leq \sum_{k=0}^n \|d_k(A; V^*, V)\| \|x_k\| \\ &\leq \|A\| \left(\sum_{k=0}^{\infty} \frac{(k+1)^2}{2^k} \right)^{1/2} \left(\sum_{k=0}^n \|x_k\|^2 \right)^{1/2}. \end{aligned}$$

Applying Lemma 2, we derive the existence of Z .

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