ON MAXIMALITY OF GORENSTEIN SEQUENCES

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ABSTRACT. It is well known that if A is a Gorenstein ring, then every ideal generated by a regular sequence $\mathbf{x} \subset A$ has irreducible (minimal) primary components. This feature led us to define a Gorenstein sequence of a ring A to be any ordered regular sequence $\mathbf{x} = \{x_1, \dots, x_r\} \subset A$ such that for every $i \in \{1, \dots, r\}$ the ideal $(x_1, \dots, x_i)$ has irreducible minimal primary components. We showed for Gorenstein sequences (**G-sequences** for short) some parallels of well-known properties of regular sequences and moreover by means of **G-sequences** we gave the following natural characterization of local Gorenstein rings: "A local ring $(A, \mathfrak{m})$ is Gorenstein iff $\mathfrak{m}$ contains a **G-sequence** of length $= K - \dim A$ ".

In this note we are going to give some information about "maximality" of **G-sequences** in a local ring A , producing sufficient conditions on A in order that the maximal **G-sequences** of A all have the same length, i.e. in order to give a "good" definition of **G-depth** A . Furthermore, we will state some results about the **G-depth** behavior with respect to local flat ring homomorphisms.

0. Notations and preliminary results. Throughout this paper A will denote a commutative noetherian ring with unity. In addition, we will say that the Gorenstein locus of A is open if there exists an ideal $\mathfrak{S} \subset A$ such that for every $\mathfrak{p} \in \text{Spec } A$, $A_{\mathfrak{p}}$ is a local Gorenstein ring iff $\mathfrak{p} \not\supset \mathfrak{S}$. In [H. K., Vortrag 6] it has been shown that the Gorenstein locus of a local Cohen-Macaulay (C. M. for short) ring A is open if there exists the canonical module K_A (for the definition and properties of K_A see [H. K., Vortrag 5] where it has also been given a sufficient criterion for the existence of K_A). If K_A exists then the ideal $\mathfrak{S} \subset A$ which defines the non-Gorenstein locus of A admits an explicit description (see [H.K., Bemerkung and Lemma 6.19]) as the ideal generated by the elements $\varphi(x)$ with $x \in K_A$, $\varphi \in \text{Hom}_A(K_A, A)$, i.e. $\mathfrak{S}$ is the trace ideal of K_A (cf. [De M. I., Chapter 1, §1B]).

In this connection we have the following handy characterization:

0.1. If A is a ring with open Gorenstein locus and $\mathfrak{S} \subset A$ is the ideal defining the non-Gorenstein locus of A , then an ordered regular sequence $\{x_1, \dots, x_r\} \subset A$ is a **G-sequence** iff for every $i \in \{1, \dots, r\}$ and $\mathfrak{p} \in \text{Ass}(A/(x_1, \dots, x_i))$ we have $\mathfrak{p} \not\supset \mathfrak{S}$.

Now we recall explicitly two concepts which will be frequently used in the following.

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0.2. The $\mathbf{G}$ -depth of a local ring $(A, \mathfrak{m})$ is the largest number of elements in $\mathfrak{m}$ making up a $\mathbf{G}$ -sequence of A . If $\mathfrak{m}$ does not contain any $\mathbf{G}$ -sequence of positive length, we define the $\mathbf{G}$ -depth A to be 0 if the zero ideal of A has irreducible minimal primary components (i.e. the empty set $\emptyset \subset A$ makes up a $\mathbf{G}$ -sequence of length 0), otherwise $\mathbf{G}$ -depth $A = -\infty$ (cf. [$\mathbf{M}_1$, Definition 4.1]).

0.3. A ring A is said to be $\mathbf{G}_n$ if $\mathbf{G}$ -depth $A_{\mathfrak{p}} \geq \min(n, \text{ht } \mathfrak{p})$ for every $\mathfrak{p} \in \text{Spec } A$ (cf. [$\mathbf{M}_1$, Definition 4.5.]).

In [$\mathbf{M}_2$, Theorem 2.1] we gave several equivalent characterizations for the $\mathbf{G}_n$ condition, showing in particular that a ring A is $\mathbf{G}_n$ iff it is S_n (i.e. depth $A_{\mathfrak{p}} \geq \min(n, \text{ht } \mathfrak{p})$ for every $\mathfrak{p} \in \text{Spec } A$) and $A_{\mathfrak{p}}$ is a local Gorenstein ring for every $\mathfrak{p} \in \text{Spec } A$ such that $\text{ht } \mathfrak{p} \leq n$. Here we can add the following:

0.4. If A is an S_n ring with open Gorenstein locus and $\mathfrak{S} \subset A$ is the ideal defining the non-Gorenstein locus of A , then A is $\mathbf{G}_n$ iff $\text{ht } \mathfrak{S} > n$.

1. Principal results and examples.

PROPOSITION 1.1. *Let A be any local C. M. ring with open Gorenstein locus and let $\mathfrak{S} \subset A$ be the ideal which defines the non-Gorenstein locus of A ; then:*

- (a) *if $\text{ht } \mathfrak{S} = 0$, A does not contain any $\mathbf{G}$ -sequence;*
- (b) *if $\text{ht } \mathfrak{S} > 0$, all maximal $\mathbf{G}$ -sequences of A have the same length (exactly equal to $K - \dim A$ or $K - \dim A - 1$, according as A is Gorenstein or not).*

PROOF. (a) A is not a $\mathbf{G}_0$ ring (cf. 0.4) so it is clear that A does not contain any $\mathbf{G}$ -sequence of length $= 0$ (cf. [$\mathbf{M}_2$, Osservazione (iii)]). On the other hand one can easily see that A does not even contain $\mathbf{G}$ -sequences of positive length since if there existed $\mathbf{x} \subset A$ a $\mathbf{G}$ -sequence of length $= r > 0$, then both $A/(\mathbf{x})$ and A would be $\mathbf{G}_0$ rings (respectively by [$\mathbf{M}_2$, Osservazione (iv)] and [$\mathbf{M}_3$, Lemma 2.2]) and this would clearly give a contradiction.

(b) First of all we observe that if A is a Gorenstein ring, then manifestly all maximal $\mathbf{G}$ -sequences of A must have the same length exactly equal to $K - \dim A$ since it follows directly from the definitions that all regular sequences of any (not necessarily local) Gorenstein ring are also $\mathbf{G}$ -sequences (cf. [$\mathbf{B}$, Fundamental Theorem], and [$\mathbf{M}_1$, Definition 2.1]). Therefore to complete our proof we have only to examine the case $\mathfrak{S} \subsetneq A$. Here $\text{ht } \mathfrak{S} > 0$ implies that for every $\mathfrak{p} \in \text{Spec } A$ such that $\text{ht } \mathfrak{p} = 0$, $A_{\mathfrak{p}}$ is a local Gorenstein ring (cf. 0.4): hence all (minimal) primary components of the zero ideal in A are irreducible, i.e. the empty set $\emptyset \subset A$ is a $\mathbf{G}$ -sequence of length $= 0$ (cf. [$\mathbf{M}_2$, Osservazione (vi)]). This just proves our theorem in case $K - \dim A = 1$ where $\emptyset$ is actually the only (maximal) $\mathbf{G}$ -sequence of A (we are assuming A is not Gorenstein), so from now on we can suppose $K - \dim A > 1$. Since A is a C. M. ring $\text{ht } \mathfrak{S} > 0$ implies also that there exists some element $i \in \mathfrak{S}$ which is regular in A and so can be completed to a maximal regular sequence of A , say $\mathbf{x} = \{i, x_2, \dots, x_m\}$ with $m = K - \dim A$. We want to show that $\mathbf{x}' = \{x_2, \dots, x_m\}$ is a $\mathbf{G}$ -sequence of A ; in this connection to say that $\mathbf{x} \subset A$ is a regular sequence means in particular that $i \notin \mathfrak{P}'$ for any $\mathfrak{P}' \in \text{Ass } (A/(\mathbf{x}'))$, i.e. $A_{\mathfrak{P}'}$ is a local Gorenstein ring for every $\mathfrak{P}' \in \text{Ass } (A/(\mathbf{x}'))$ and this means that $\mathbf{x}'$ is a $\mathbf{G}$ -sequence, so $\emptyset$ is not a maximal $\mathbf{G}$ -sequence in A . Then let $\mathbf{y} \subset A$ be any maximal $\mathbf{G}$ -sequence of positive length $= s < m$

(recall that we are assuming A is not Gorenstein): for every $\mathfrak{Q} \in \text{Ass}(A/(\mathbf{y}))$ clearly $\mathfrak{Q} \not\supset \mathfrak{F}$ so there exists some element $j \in \mathfrak{F}$ which is a non-zero-divisor modulo $(\mathbf{y})$ such that $\{\mathbf{y}, j\}$ is a regular sequence of length $= s + 1$. If $s < m - 1$, $\{\mathbf{y}, j\}$ can be completed to a maximal regular sequence of A , say $\{\mathbf{y}, j, y_{s+2}, \dots, y_m\}$; here applying the same argument as above we can see that $\{\mathbf{y}, y_{s+2}, \dots, y_m\}$ is a $\mathbf{G}$ -sequence of A containing $\mathbf{y}$, contradicting the maximality of $\mathbf{y}$, so all maximal $\mathbf{G}$ -sequences of A actually have length $= K - \dim A - 1$.

COROLLARY 1.2. *If A is a local C. M. ring with open Gorenstein locus, then $\mathbf{G}$ -depth A is the length of any maximal $\mathbf{G}$ -sequence in A (as usual $\mathbf{G}$ -depth $A = -\infty$ if A does not contain $\mathbf{G}$ -sequences of any length).*

COROLLARY 1.3. *Let A be any S_n local ring with $K - \dim A > n > 1$. If the Gorenstein locus of A is open, then either A does not contain any $\mathbf{G}$ -sequence or all maximal $\mathbf{G}$ -sequences of A have length $\geq n - 1$.*

PROOF. Let $\mathfrak{F}$ be the ideal which defines the non-Gorenstein locus of A ; we can observe that, as in Proposition 1.1, our conclusion and proof depend on height $\mathfrak{F}$. Precisely: $\text{ht } \mathfrak{F} = 0$ implies both A does not contain any $\mathbf{G}$ -sequence of length $= 0$ (in fact in this case A is not $\mathbf{G}_0$) and A does not contain any $\mathbf{G}$ -sequence of positive length (namely if $\mathbf{x} = \{x_1, \dots, x_r\}$ would be a $\mathbf{G}$ -sequence of length $= r > 0$, then for all $i \in \{1, \dots, r\}$ x_i would be a $\mathbf{G}$ -sequence which generates an unmixed ideal (cf. [S, Theorem 2.2]), so $A/(x_i)$ and then A (by [R. F., Proposition 3]) would be S_1 and $\mathbf{G}_0$ rings contradicting the fact that A is not $\mathbf{G}_0$).

$\text{ht } \mathfrak{F} > 0$ implies $\emptyset$ is not a maximal $\mathbf{G}$ -sequence (we can use an argument like that of Proposition 1.1). Then let $\mathbf{x} \subset A$ be any maximal $\mathbf{G}$ -sequence of length $= s > 0$, if $s < n - 1$ for every $\mathfrak{P} \in \text{Ass}(A/(\mathbf{x})), \mathfrak{P} \not\supset \mathfrak{F}$, $(\mathbf{x})$ is an unmixed ideal by [S, Theorem 2.2], so there exists some element $j \in \mathfrak{F}$ which is a non-zero-divisor modulo $(\mathbf{x})$. Considering the regular sequence $\{\mathbf{x}, j\}$, we can conclude, as in Proposition 1.1:

REMARK I. From [M₃, Lemma 2.2] and [R. F., Proposition 3], we can deduce some information about the existence of $\mathbf{G}$ -sequences in a local ring A without having resort to the hypothesis that the Gorenstein locus of A is open; precisely we can say:

(i) In any local non- $\mathbf{G}_0$ ring A which satisfies the “saturated chain condition on prime ideals”, there exist no $\mathbf{G}$ -sequences.

(ii) In any local non- $\mathbf{G}_0$ ring A there exist no $\mathbf{G}$ -sequences (of any length) which generated unmixed ideals, but a priori we do not have any information about possible $\mathbf{G}$ -sequences which generated mixed ideals.

REMARK II. From the proof of Corollary 1.3 we can deduce that in a local S_n ring A ($K - \dim A > n \geq 1$) which is $\mathbf{G}_0$ and has open Gorenstein locus, there exist $\mathbf{G}$ -sequences of length $= \text{depth } A - 1$ but a priori we cannot say if this must be the length of every maximal $\mathbf{G}$ -sequence of A , so actually we do not know whether there may exist maximal $\mathbf{G}$ -sequences of different lengths in A .

Recall explicitly the following notation introduced in [W.I.T.O., Definition 1.7].

DEFINITION 1.4. A ring homomorphism $\varphi: A \rightarrow B$ is Gorenstein if it is flat and has Gorenstein fibres.

LEMMA 1.5. *Let A be any ring with open Gorenstein locus. Then, for every Gorenstein homomorphism $\varphi: A \rightarrow B$, the Gorenstein locus of B is open.*

PROOF. Let $f: Y = \text{Spec } B \rightarrow \text{Spec } A = X$ be the induced morphism and let $U \subset X$ be the Gorenstein locus of A . For every $\mathfrak{P} \in f^{-1}(U)$, $B_{\mathfrak{P}}$ is Gorenstein; namely, putting $\mathfrak{p} = f(\mathfrak{P})$, clearly $\mathfrak{p} \in U$ so the local homomorphism $\psi: A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ (induced by φ) is flat with $A_{\mathfrak{p}}$ Gorenstein (since $\mathfrak{p} \in U$) and $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$ Gorenstein (since it is a localization of the fibre of φ at $\mathfrak{p}$). On the other hand, for every $\mathfrak{Q} \in Y - f^{-1}(U)$, $B_{\mathfrak{Q}}$ is not Gorenstein since putting $\mathfrak{q} = \mathfrak{Q} \cap A$ clearly $\mathfrak{q} \notin U$, so $A_{\mathfrak{q}}$ is not Gorenstein. Therefore the Gorenstein locus of B is precisely $f^{-1}(U)$ which, by the hypothesis, is clearly open.

PROPOSITION 1.6. *Let $(A, \mathfrak{m})$ be any local C. M. ring with open Gorenstein locus. Then for every local Gorenstein homomorphism $\varphi: A \rightarrow B$, we have*

$$\mathbf{G} - \text{depth } B = \mathbf{G} - \text{depth } A + \mathbf{G} - \text{depth } B/\mathfrak{m}B.$$

PROOF. Observe that under the given hypotheses not only B is a (local) C. M. ring (cf. [D, Corollary 5.1]) but also its Gorenstein locus is open (cf. Lemma 1.5): therefore $\mathbf{G} - \text{depth } B$ is actually well defined (cf. Corollary 1.2). In addition, we notice that if A is Gorenstein, then there is nothing to prove, since in that case B is a (local) Gorenstein ring by [W.I.T.O., Theorem 1], so clearly

$$\begin{aligned} \mathbf{G} - \text{depth } B &= K - \dim B = K - \dim A + K - \dim B/\mathfrak{m}B \\ &= \mathbf{G} - \text{depth } A + \mathbf{G} - \text{depth } B/\mathfrak{m}B. \end{aligned}$$

Then, to show our contention, we only have to examine the case A is not Gorenstein which, according to Proposition 1.1, splits into $\mathbf{G} - \text{depth } A = -\infty$ and $\mathbf{G} - \text{depth } A = K - \dim A - 1$ (automatically ≥ 0). If $\mathbf{G} - \text{depth } A = -\infty$, then we cannot have $\mathbf{G} - \text{depth } B \geq 0$, since that would mean B is at least $\mathbf{G}_0$ and hence also A would be at least $\mathbf{G}_0$ (cf. [M₁, Theorem 5.1]), contradicting $\mathbf{G} - \text{depth } A = -\infty$; then

$$\begin{aligned} \mathbf{G} - \text{depth } B &= -\infty = -\infty + K - \dim B/\mathfrak{m}B \\ &= \mathbf{G} - \text{depth } A + \mathbf{G} - \text{depth } B/\mathfrak{m}B. \end{aligned}$$

If $0 \leq \mathbf{G} - \text{depth } A = K - \dim A - 1$, then B is a $\mathbf{G}_0$ ring (cf. [M₁, Corollario 5.2]), that is $\mathbf{G} - \text{depth } B \geq 0$. Moreover, since we are assuming A is not Gorenstein, we cannot have $\mathbf{G} - \text{depth } B = K - \dim B$ (cf. [W.I.T.O., Theorem 1]); thus again

$$\begin{aligned} \mathbf{G} - \text{depth } B &= K - \dim B - 1 = K - \dim A - 1 + K - \dim B/\mathfrak{m}B \\ &= \mathbf{G} - \text{depth } A + \mathbf{G} - \text{depth } B/\mathfrak{m}B. \end{aligned}$$

REMARK III. Observe that if the local C. M. ring $(A, \mathfrak{m})$ has the canonical module K_A , then every local flat ring homomorphism $\varphi: A \rightarrow B$ such that

$B/\mathfrak{m}B$ is Gorenstein (i.e. $K_B \simeq K_A \otimes_A B$ (cf. [H.K., Satz 6.14.])) is actually a Gorenstein homomorphism. Namely, using the same notations as in Lemma 1.5, for any $\mathfrak{p} \in X$, the fibre of φ at $\mathfrak{p}$ is Gorenstein since (cf. [E.G.A., IV₂, Lemma 7.3.2]) for every $\mathfrak{P} \in f^{-1}(\mathfrak{p})$, $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$ is Gorenstein, being

$$\begin{aligned} (K_{A_{\mathfrak{p}}}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}} &= (K_A)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}} = K_A \otimes_A A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}} = K_A \otimes_A B_{\mathfrak{P}} \\ &= K_A \otimes_A B \otimes_B B_{\mathfrak{P}} = K_B \otimes_B B_{\mathfrak{P}} = (K_B)_{\mathfrak{P}} = K_{B_{\mathfrak{P}}} \end{aligned}$$

(cf. [R, (3) Theorem] and [H.K., Korollar 5.25]). Notice that the hypothesis “ $B/\mathfrak{m}B$ is Gorenstein” cannot be avoided; namely, we can easily see that Proposition 1.6 does not hold for a local flat homomorphism $\varphi: A \rightarrow B$ of complete, equidimensional local rings such that A is Gorenstein and B is $\mathbf{G}_0$ but not Gorenstein (e.g., fix a field k , take

$$A = k[[U^N]] \quad (N = \text{l.c.m.}(6, 7, 8, 9)),$$

$$B = k[[x, y, z, t]]/(t^2 - x^3, z^2 - yt, y^2 - xz, yz - xt),$$

and φ the inclusion map (which is clearly flat and local); what we get is $\mathbf{G}$ -depth $A = 1$, $\mathbf{G}$ -depth $B = 0$ (B is a 1-dimensional complete integral domain which is not Gorenstein (cf. [K, Theorem]), and $\mathbf{G}$ -depth $B/\mathfrak{m}B = -\infty$).

Finally we are going to list explicitly some examples of local flat ring homomorphisms for which Proposition 1.6 holds.

1.7. Let $(A, \mathfrak{m})$ be any local C.M. ring with residue field k such that K_A exists. Then:

(i) If x is an indeterminate, for every maximal ideal $\mathfrak{M} \subset A[x]$ such that $\mathfrak{M} \cap A = \mathfrak{m}$, we have $\mathbf{G}$ -depth $A[x]_{\mathfrak{M}} = \mathbf{G}$ -depth $A + 1$ (in fact the fibre $k \otimes_A A[x]_{\mathfrak{M}}$ is isomorphic to the 1-dimensional Gorenstein ring $k[x]$ localized at the maximal ideal $\mathfrak{M} = \mathfrak{M}k[x]$ (cf. [G.S., Example 12.1])).

(ii) If F is a finite abelian group, for every maximal ideal $\mathfrak{M} \subset A[F]$, we have $\mathbf{G}$ -depth $A[F]_{\mathfrak{M}} = \mathbf{G}$ -depth A (in fact $k \otimes_A A[F]_{\mathfrak{M}}$ is isomorphic to the 0-dimensional Gorenstein ring $k[F]$ localized at the maximal ideal $\mathfrak{M} = \mathfrak{M}k[F]$ (cf. [P, Corollaire 2])).

(iii) If hA is the henselization of A with respect to $\mathfrak{m}$, we have $\mathbf{G}$ -depth ${}^hA = \mathbf{G}$ -depth A (in fact $k \otimes_A {}^hA \simeq {}^hA/\mathfrak{m}{}^hA \simeq k$ (cf. [E.G.A., IV₄, Theorem 18.6.6])).

(iv) If $\hat{A}$ is the $\mathfrak{m}$ -adic completion of A , we have $\mathbf{G}$ -depth $\hat{A} = \mathbf{G}$ -depth A (in fact $k \otimes_A \hat{A} \simeq \hat{A}/\mathfrak{m}\hat{A} \simeq k$ (cf. [D, §6.A5])).

(v) If $A[[x]]$ is the formal power series ring (in one indeterminate) over A , we have $\mathbf{G}$ -depth $A[[x]] = \mathbf{G}$ -depth $A + 1$ (in fact the fibres of $A \rightarrow A[[x]]$ are canonically isomorphic to the fibres of $B \rightarrow B[[x]]$ (where B is a local Gorenstein ring such that $A = B/\mathfrak{b}$ (cf. [R, (3) Theorem])) at the prime ideals of B containing $\mathfrak{b}$, and $B \rightarrow B[[x]]$ is a local flat ring homomorphism whose fibre at the closed point of $\text{Spec } B$ is equal to $k \otimes_B B[[x]]$ which is clearly a (local) 1-dimensional Gorenstein ring (cf. [G.S., Theorem 9.8] and [W.I.T.O., Theorem 2])).

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