INERT EXTENSIONS OF KRULL DOMAINS

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ABSTRACT. Let $A \subseteq B$ be integral domains with B an inert extension of a Krull domain A . Let $\mathcal{P}(A)$ be the set of height one primes of A , and let $T = \bigcap_{p \in \mathcal{P}(A)} B \otimes_A A_p$. When each $B_p = B \otimes_A A_p$ is a UFD, a necessary and sufficient condition for T to be a Krull domain is obtained. If T is a Krull domain and each B_p is a UFD, then the divisor class groups of A and T are isomorphic under the natural mapping.

These results are applied to $A \subseteq B$ when B is a symmetric algebra over A and when B is locally a polynomial ring over A .

Let $A \subseteq B$ be integral domains. Following Cohn [3] we say that B is an inert extension of A if for any nonzero $x, y \in B$, $xy \in A$ implies $x \in A$, $y \in A$. If B is an inert extension of A and a Krull domain, then A is easily seen to be a Krull domain. In this paper we give a partial converse to this result, and we compare the class groups of A and B , when both are Krull domains.

This paper originated as a study of extensions $A \subseteq B$ with A a Krull domain and B a symmetric algebra over A . Under these hypotheses it is known that when A is a UFD, B is a UFD if and only if B is a factorial A -module [4], [5]. An A -module M over a UFD A is factorial if it is torsion-free and every element $x \in M$ may be written uniquely $x = ax^*$, with $a \in A$ and x^* an element of M divisible only by units of A . Elements of M divisible only by units are called primitive.

We generalize the idea of a factorial module as follows. Let A be a Krull domain, M an A -module, and $\mathcal{P}(A)$ the set of height one primes of A . We say that M is a Krull A -module if M is torsion-free, $M_p = M \otimes_A A_p$ is a factorial A_p -module for each $p \in \mathcal{P}(A)$, $M = \bigcap_{p \in \mathcal{P}(A)} M_p$, and each nonzero x in M is primitive in all but a finite number of M_p (i.e. the intersection is "locally finite"). Our main result, Theorem 1.6, shows that if $A \subseteq B$ is an inert extension with A a Krull domain and B_p a UFD for each $p \in \mathcal{P}(A)$, then $T = \bigcap_{p \in \mathcal{P}(A)} B_p$ is a Krull domain if and only if it is a Krull A -module, in which case $\text{Cl}(A) \cong \text{Cl}(T)$. In the second section we show that in the case when B is a symmetric algebra over A , the condition that B_p be a UFD for each $p \in \mathcal{P}(A)$ is superfluous. We conclude with some examples.

We owe a debt of originality to §10 of [7], as our main results are generalizations of Theorem 10.11 of [7] and our arguments, for the most part,

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are adaptations of those of [7]. Indeed, one might consider this paper a supplement to [7].

1. For a Krull domain A let $\text{Div}(A)$ be the group of divisorial fractional ideals of A , $\text{Prin}(A)$ the group of principal fractional ideals of A , $\text{Cl}(A) = \text{Div}(A)/\text{Prin}(A)$ the divisor class group of A , and $\mathfrak{P}(A)$ the set of height one prime ideals of A . If $A \subseteq B$ are Krull domains we say that (PDE) holds if $P \in \mathfrak{P}(B)$ implies $P \cap A \in \mathfrak{P}(A) \cup \{0\}$. When (PDE) holds, the map $\text{Div}(A) \rightarrow \text{Div}(B)$ given by $\mathfrak{A} \mapsto B: (B: \mathfrak{A}B)$ is a homomorphism [7, Proposition 6.3(d)] which clearly maps $\text{Prin}(A)$ into $\text{Prin}(B)$, and so induces a natural homomorphism $\text{Cl}(A) \rightarrow \text{Cl}(B)$. We begin by investigating this map when $A \subseteq B$ is an inert extension.

LEMMA 1.1. *Let $A \subseteq B$ be an inert extension of integral domains and let $\mathfrak{A}$ be an ideal of A . Then $A: (A: \mathfrak{A}) \supseteq (B: (B: \mathfrak{A}B)) \cap A$.*

PROOF. Let $x \in A$ and suppose $x(B: \mathfrak{A}B) \subseteq B$. If $y \in A: \mathfrak{A}$, then $y \in B: \mathfrak{A}B$ and hence $xy \in B$. Say $xy = b$ and $y = a/t$ with $a, t \in A$. Then $tb = ax \in A$, so $b \in A$ and hence $x \in A: (A: \mathfrak{A})$.

PROPOSITION 1.2. *Let $A \subseteq B$ be an inert extension of Krull domains for which (PDE) holds. Then $\text{Cl}(A) \rightarrow \text{Cl}(B)$ is injective.*

PROOF. We first show that $\text{Div}(A) \rightarrow \text{Div}(B)$ is injective. Since $\text{Div}(A)$ and $\text{Div}(B)$ are free abelian groups on the sets $\mathfrak{P}(A)$ and $\mathfrak{P}(B)$, it suffices to show that for each $p \in \mathfrak{P}(A)$ there is a $P \in \mathfrak{P}(B)$ such that $P \cap A = p$. But if p is a prime divisorial ideal of A , then $p = (B: (B: pB)) \cap A$ by the lemma. Hence $B: (B: pB)$ is a proper divisorial ideal of B and is contained in some $P \in \mathfrak{P}(B)$. Then $P \cap A = p$.

It remains to show that $\text{Prin}(B) \cap \text{Div}(A) = \text{Prin}(A)$. But if $\mathfrak{A}$ is a divisorial ideal of A such that $B: (B: \mathfrak{A}B)$ is principal, then from $\mathfrak{A} = (B: (B: \mathfrak{A}B)) \cap A$ and the inertness of the extension we deduce that $\mathfrak{A}$ is principal.

LEMMA 1.3. *Let $A \subseteq B$ be an extension of Krull domains for which (PDE) holds. If $\mathfrak{B}$ is a divisorial ideal in B then $\mathfrak{B} = \bigcap_{p \in \mathfrak{P}(A)} \mathfrak{B}_p$.*

PROOF. Suppose that $\mathfrak{B} \neq \bigcap_{p \in \mathfrak{P}(A)} \mathfrak{B}_p$. For each element $x \in \bigcap_{p \in \mathfrak{P}(A)} \mathfrak{B}_p \setminus \mathfrak{B}$, $(\mathfrak{B}:_B x) = \{y \in B \mid yx \in \mathfrak{B}\}$ is a divisorial ideal of B . Choose such an x so that $(\mathfrak{B}:_B x)$ is maximal in this family of divisorial ideals. To see that $(\mathfrak{B}:_B x)$ is prime let $f, g \in B$ with $fg \in (\mathfrak{B}:_B x)$. If $g \notin (\mathfrak{B}:_B x)$, then $gx \in \bigcap_{p \in \mathfrak{P}(A)} \mathfrak{B}_p \setminus \mathfrak{B}$ and $(\mathfrak{B}:_B gx) \supseteq (\mathfrak{B}:_B x) + fB$. By the maximality of $(\mathfrak{B}:_B x)$ we have $f \in (\mathfrak{B}:_B x)$. Let $P = (\mathfrak{B}:_B x)$. Being divisorial, P is a height one prime of B and hence there is a height one prime p of A such that $P \cap A \subseteq p$. Then $P_p \neq B_p$, but $P_p = (\mathfrak{B}:_B x)_p = (\mathfrak{B}_p:_{B_p} x) = B_p$, a contradiction. This proves the lemma.

THEOREM 1.4. *Let $A \subseteq B$ be an inert extension of Krull domains for which (PDE) holds. Further suppose that B_p is a UFD for each $p \in \mathfrak{P}(A)$. Then the*

natural homomorphism $\text{Cl}(A) \rightarrow \text{Cl}(B)$ *is an isomorphism.*

PROOF. In virtue of Proposition 1.2, we only need to prove that $\text{Cl}(A) \rightarrow \text{Cl}(B)$ is surjective. By Nagata's Theorem [7, Theorem 7.1] it is sufficient to show that if $p \in \mathfrak{P}(A)$, then the divisorial ideal $B: (B: pB)$ is prime in B . Let $q \in \mathfrak{P}(A)$. Then $(B: (B: pB))_q = B_q: (B_q: pB_q)$; but pB_q is principal so that $B_q: (B_q: pB_q) = pB_q$. By the lemma,

$$B: (B: pB) = \bigcap_{q \in \mathfrak{P}(A)} (B: (B: pB))_q = \bigcap_{q \neq p} B_q \cap pB_p = B \cap pB_p$$

which is prime since B_p is a UFD.

In the next section we give an example to show that the hypothesis that B_p be a UFD for $p \in \mathfrak{P}(A)$ cannot be deleted from Theorem 1.4.

To prepare for our next result we first need some equivalent forms of (PDE).

PROPOSITION 1.5. *Let $A \subseteq B$ be Krull domains. The following are equivalent:*

- (1) $A \subseteq B$ satisfies (PDE).
- (2) $B = \bigcap_{p \in \mathfrak{P}(A)} B_p$.
- (3) Every A -sequence a, b is a B -sequence.

PROOF. We first show that (2) and (3) are equivalent with B replaced by any torsion-free A -module M .

(2) $\Rightarrow$ (3). Let a, b be an A -sequence and let $x, y \in M$ be such that $ax = by$. Given $p \in \mathfrak{P}(A)$, we have $(a, b) \not\subseteq p$ since p is divisorial. If $b \notin p$ then $y \in aM_p$; and if $a \notin p$ then $y \in M_p = aM_p$. Hence

$$y \in \bigcap_{p \in \mathfrak{P}(A)} aM_p = a \bigcap_{p \in \mathfrak{P}(A)} M_p = aM.$$

Therefore a, b is an M -sequence.

(3) $\Rightarrow$ (2). Let $x \in \bigcap M_p$. Then for each $p \in \mathfrak{P}(A)$, $(M: x)_p = (M_p: x) = D_p$, so $(M: x) \not\subseteq p$. Let $a \in (M: x)$, $a \neq 0$, and let $p_1, \dots, p_n$ be the height one primes containing a . Then $aA = \bigcap_{i=1}^n Q_i$ with Q_i p_i -primary for $i = 1, \dots, n$ [8, Corollary 43.10]. Denoting by $Z(A/aA)$ the set of zero-divisors on the module A/aA , we now have $Z(A/aA) \subseteq \bigcup_{i=1}^n p_i$ [1, Proposition 4.7]. Since $(M: x) \not\subseteq p_i$ for any i , we may choose $b \in (M: x) \setminus Z(A/aA)$. Then a, b is an A -sequence and therefore an M -sequence. Since $b(ax) = a(bx)$ with $ax, bx \in M$ we have $ax \in aM$ and hence $x \in M$. Thus $M = \bigcap M_p$.

(3) $\Rightarrow$ (1). Let $P \in \mathfrak{P}(B)$ and set $p = P \cap A$. If $\text{ht } p > 1$, pick $a \in p$, $a \neq 0$ and then choose $b \in p \setminus Z(A/aA)$ as in the proof of (3) $\Rightarrow$ (2). Then a, b is an A -sequence and therefore a B -sequence in P , so $\text{ht } P \geq 2$ [9, Theorem 131]. This contradicts $\text{ht } P = 1$, so $\text{ht } p \leq 1$.

(1) $\Rightarrow$ (3). Let a, b be an A -sequence. Since B is a Krull domain, $aB = \bigcap_{i=1}^n Q_i$ with each Q_i P_i -primary for some $P_i \in \mathfrak{P}(B)$. Hence $Z(B/aB) \subseteq \bigcup_{i=1}^n P_i$. If $b \in Z(B/aB)$ we get $a, b \in P_i$ for some i . But then $a, b \in P_i \cap A$ which implies that $\text{ht}(P_i \cap A) \geq 2$, contradicting (PDE). It follows that $b \notin Z(B/aB)$, i.e. that a, b is a B -sequence.

Now for the main theorem.

THEOREM 1.6. *Let $A \subseteq B$ be an inert extension with A a Krull domain and B_p a UFD for each $p \in \mathfrak{P}(A)$. Let $T = \bigcap B_p$. Then $A \subseteq T$ is an inert extension and T is a Krull domain if and only if it is a Krull A -module, in which case $\text{Cl}(T) \cong \text{Cl}(A)$, by the natural homomorphism.*

PROOF. It is easy to see that $A \subseteq T$ is an inert extension. And if T is a Krull domain then since $T = \bigcap T_p$, (PDE) holds for the extension $A \subseteq T$ by Proposition 1.5. The desired isomorphism of class groups now follows from Theorem 1.4. Thus it only remains to demonstrate the equivalence in the present theorem.

Suppose T is a Krull domain. For $p \in \mathfrak{P}(A)$, $T_p = B_p$ is a UFD and an inert extension of the DVR A_p . Hence T_p is a factorial A_p -module. It also follows that pT_p is a height one prime of T_p , and so $pT_p \cap T$ is height one. It follows that if $f \in T$, $f \neq 0$, then f belongs to only a finite number of the ideals $pT_p \cap T$. Thus f is primitive as an element of the A_p -module T_p for all but a finite number of p in $\mathfrak{P}(A)$.

Conversely, suppose that T is a Krull A -module. Since pT_p is a principal prime of the UFD $T_p = B_p$, $T_p = (T_p)_{pT_p} \cap T_{(A \setminus \{0\})}$. Thus

$$T = \bigcap_{p \in \mathfrak{P}(A)} (T_p)_{pT_p} \cap T_{(A \setminus \{0\})}.$$

For each element x in T , x is primitive in all but a finite number of the T_p implies that x is a unit in all but a finite number of the $(T_p)_{pT_p}$. Hence by [14, Proposition 4.1], $\bigcap (T_p)_{pT_p}$ is a Krull domain and, since $T_{(A \setminus \{0\})}$ is a UFD, T is a Krull domain.

REMARKS. (1) In the next section we will give an example to show that, under the hypotheses of Theorem 1.6, B may fail to be a Krull domain even though T is one.

(2) If in Theorem 1.6, B is a Krull domain, then T is a subintersection of B and hence is a Krull domain. In fact,

$$T = \bigcap_{p \in \mathfrak{P}(A)} B_p = \bigcap_{p \in \mathfrak{P}(A)} \bigcap_{P \in \mathfrak{P}(B); P \cap A \subseteq p} B_P = \bigcap_{\text{ht}(P \cap A) \leq 1; P \in \mathfrak{P}(B)} B_P.$$

By Nagata's theorem [7, Theorem 7.1] there is an exact sequence of abelian groups $0 \rightarrow K \rightarrow \text{Cl}(B) \rightarrow \text{Cl}(T) \rightarrow 0$, where K is generated by the divisor classes of height one primes P of B such that $\text{ht}(P \cap A) > 1$. Note that this relates $\text{Cl}(A)$ and $\text{Cl}(B)$, since $\text{Cl}(T) \cong \text{Cl}(A)$. Furthermore, $K = 0$ if and only if every $P \in \mathfrak{P}(B)$ such that $\text{ht}(P \cap A) > 1$ is principal. But if P is principal in B and $P \cap A \neq 0$ any generator for P must lie in A , whence $P \cap A$ is principal and has height one. It follows that $K = 0$ if and only if (PDE) holds for $A \subseteq B$, i.e. if and only if $B = T$.

(3) Eakin and Silver [6] examine extensions $A \subseteq B$ of integral domains in which B is locally a polynomial ring over A . Since inertness is a local property, all such extensions are inert. If, moreover, A is a Krull domain, $B = T$ [6, (2.26)] and B_p being a polynomial ring over A_p is a UFD. From these

observations one easily obtains [6, (3.1)] and the equivalence of (i) and (iii) in [6, (2.27)] as corollaries to Theorem 1.6.

2. Let A be a Krull domain with quotient field k and let M be a torsion-free A -module. Form the symmetric algebra $S_A(M)$. For each $p \in \mathcal{P}(A)$, $S(M)_p = S_{A_p}(M_p)$ is a graded subring of the polynomial ring $S_k(M \otimes k)$, since M_p is flat over A_p . Letting $T = \bigcap_{p \in \mathcal{P}(A)} S(M)_p$, we find that T is a graded domain with A as its homogeneous component of degree zero. $A \subseteq T$ is thus an inert extension.

PROPOSITION 2.1. *In the notation of the above paragraph, if T is either a Krull domain or a Krull A -module, then $T_p = S(M)_p$ is a UFD for each $p \in \mathcal{P}(A)$.*

PROOF. $S(M)_p$ is a UFD if and only if it is a factorial A_p -module. This holds trivially if T is a Krull A -module. If T is a Krull domain, then $S(M)_p = T_p$ satisfies the ascending chain condition on principal ideals (a.c.c.p.). Since A_p is a DVR, $S(M)_p$ is flat over A_p and this with a.c.c.p. makes $S(M)_p$ a factorial A_p -module [5], [11].

COROLLARY 2.2. *Let A be a Krull domain, M a torsion-free A -module, and $T = \bigcap_{p \in \mathcal{P}(A)} S(M)_p$. Then T is a Krull domain if and only if it is a Krull A -module, in which case $\text{Cl}(T) \cong \text{Cl}(A)$.*

COROLLARY 2.3. *Let A be a Krull domain, M a finitely generated torsion-free A -module, and $T = \bigcap_{p \in \mathcal{P}(A)} S(M)_p$. Then T is a Krull domain and $\text{Cl}(T) \cong \text{Cl}(A)$.*

PROOF. It suffices to show that T is a Krull A -module. Each $S(M)_p$ is a polynomial ring over A_p , hence is factorial over A_p . We only need to check that the intersection $\bigcap S(M)_p$ of factorial A_p -modules is locally finite. Now we may choose a finitely generated free module F and an element $a \in A$, $a \neq 0$, such that $aF \subseteq M \subseteq F$. Then $S(aF) \subseteq T \subseteq S(F)$. $S(F)$ being a free A -module is a Krull module. Hence each nonzero element of T is primitive in $S(F)_p$ for all but a finite number of $p \in \mathcal{P}(A)$. But a is a unit in A_p for all but a finite number of $p \in \mathcal{P}(A)$ and, hence, $T_p = S(F)_p$ for all but a finite number of $p \in \mathcal{P}(A)$.

We conclude with some examples.

EXAMPLE 2.4. A symmetric algebra $B = S_A(M)$ which is a Krull domain such that (PDE) fails for $A \subseteq B$ and $\text{Cl}(A) \not\cong \text{Cl}(B)$. Let k be a field and t_1, t_2 indeterminates over k . Set

$$A = k[t_1, t_2]_{(t_1, t_2)} \quad \text{and} \quad B = A[x, y]/(t_1 x + t_2 y).$$

Then B is the symmetric algebra of the maximal ideal of A . The example is due to Samuel [13]; the fact that (PDE) fails and the class groups are not isomorphic was noted by Fossum [7, §10].

EXAMPLE 2.5. A symmetric algebra $B = S_A(M)$ over a Krull domain A such that B is an integral domain, $T = \bigcap B_p$ is a Krull domain, but B is not a Krull

domain. Let k be a field and x, y, z be indeterminates. Let $A = k[x, y]$ and let M be the ideal of A generated by x^2 and y^2 . Then $B = S_A(M) \cong A[x^2z, y^2z]$ [2], [10]. Now B is noetherian, but $xyz \notin B$ although $(xyz)^2 - (x^2z)(y^2z) = 0$. Thus B is not Krull. On the other hand, $T \cong A[z]$ is Krull.

The next example shows that we are not able to weaken the condition in Theorem 1.4 that B_p be a UFD for each $p \in \mathcal{P}(A)$.

EXAMPLE 2.6. A graded ring over a DVR which is a Krull domain and a Krull module, but not a UFD. Let V be a DVR, x and y indeterminates assigned degree 1 and $T = V[x^2, xy, y^2]$. T is a Krull V -module since it is a free V -module. To show that T is a Krull domain note that with $K[u, 1/u]$ the homogeneous quotient ring of $V[x, y]$, $T = V[x, y] \cap K[u^2, 1/u^2]$, the intersection of two Krull domains. T is not a UFD since $(xy)^2 = x^2y^2$.

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