REPRESENTATION OF NUMBERS BY CASCADES

C. C. CHEN AND D. E. DAYKIN

ABSTRACT. A cascade C is defined as a sum of binomial coefficients

$$C = \binom{a_h}{h} + \binom{a_{h-1}}{h-1} + \cdots + \binom{a_t}{t}$$

where $a_h > a_{h-1} > \cdots > a_t$. In this expression, we assume that $\binom{a}{h} = 0$ whenever $a < h$. Given a cascade C and a sequence $\varepsilon = \langle \varepsilon_h, \varepsilon_{h-1}, \dots, \varepsilon_t \rangle$ of signs (i.e. $\varepsilon_i = +1$ or -1 for each i), we define

$$\varepsilon C = \varepsilon_h \binom{a_h}{h} + \cdots + \varepsilon_t \binom{a_t}{t}.$$

Also, we put

$$\alpha C = \binom{a_h}{h+1} + \binom{a_{h-1}}{h} + \cdots + \binom{a_t}{t+1}.$$

In the paper, we prove that for any sequence $\langle n_0, n_1, \dots, n_s \rangle$ of integers, there exist a cascade C and a corresponding sequence ε of signs such that $n_i = \varepsilon \alpha^i C$ for $i = 0, 1, \dots, s$ where $\alpha^0 C = C$, $\alpha^1 C = \alpha C$, $\alpha^2 C = \alpha(\alpha^1 C)$, and recursively, $\alpha^n C = \alpha(\alpha^{n-1} C)$.

1. Introduction. Let $a_h, a_{h-1}, \dots, a_t, h, t$ be integers with $a_h > a_{h-1} > \cdots > a_t$. We shall call an expression of the form

$$(I) \quad C = \binom{a_h}{h} + \binom{a_{h-1}}{h-1} + \cdots + \binom{a_t}{t}$$

a cascade; h is called the *head*, t the *tail*, $h - t + 1$ the *length* and $a_h - h$ the *rank* of the cascade C . Note that in the expression (I), each binomial coefficient

$$\binom{a_i}{i} \quad \text{with } a_i < i \text{ or } i < 0$$

is equal to zero. Also the cascade is said to be *normal* if $a_t \geq t \geq 0$ but not $a_t = t = 0$.

For each cascade C as given in (I), we define new cascades $\alpha(C)$, $\delta(C)$, $\mu(C)$ and $\rho(C)$ respectively by:

Received by the editors May 15, 1975 and, in revised form, August 1, 1975.

AMS (MOS) subject classifications (1970). Primary 05A05, 05A10; Secondary 04A20.

© American Mathematical Society 1976

$$\begin{aligned}\alpha(C) &= \binom{a_h}{h+1} + \binom{a_{h-1}}{h} + \cdots + \binom{a_t}{t+1}, \\ \delta(C) &= \binom{a_h}{h-1} + \binom{a_{h-1}}{h-2} + \cdots + \binom{a_t}{t-1}, \\ \mu(C) &= \binom{a_h-1}{h} + \binom{a_{h-1}-1}{h-1} + \cdots + \binom{a_t-1}{t}, \\ \rho(C) &= \binom{a_h-1}{h-1} + \binom{a_{h-1}-1}{h-2} + \cdots + \binom{a_t-1}{t-1}.\end{aligned}$$

Recursively, for $n \geq 0$, $\alpha^n(C)$ means $\alpha(\alpha^{n-1}(C))$ and $\alpha^0(C) = C$. $\delta^n(C)$, $\mu^n(C)$ and $\rho^n(C)$ are defined likewise. It is evident that for any cascade C , $\delta\alpha(C) = \alpha\delta(C) = C$, $\mu(C) = \rho\alpha(C)$ and $\rho(C) = \delta\mu(C)$.

Again, let C be the cascade given in (I) and let $\varepsilon = \langle \varepsilon_h, \varepsilon_{h-1}, \dots, \varepsilon_t \rangle$ be a sequence of signs (i.e. each ε_i is either $+1$ or -1). We shall denote by εC the number

$$\varepsilon C = \varepsilon_h \binom{a_h}{h} + \varepsilon_{h-1} \binom{a_{h-1}}{h-1} + \cdots + \varepsilon_t \binom{a_t}{t}.$$

Notice that whenever we write such an expression we assume the sequence and the cascade have the same length.

As far as we know, cascades were first found by Kruskal [1] and play an important role in the theory of finite sets. They have been used [1], [2], [5] to prove the Kruskal-Katona theorem, which has the Erdős-Ko-Rado theorem as a corollary. It is well known now that given $h \geq 1$ each positive integer has a unique representation as a cascade with head h and tail > 0 . It is shown in [6] that given $c, h \geq 1$ each positive integer has a unique representation of the form $C - \delta^c(C)$, where if (I) is C then $t > 0$ and $a_i \geq 2i - c$ for $h \geq i \geq t$. Results of a different kind are obtained in [6] by attaching signs to the binomial coefficients in a cascade and then forming a family of finite sets according to certain rules. In view of all these results it is natural to study the sequence $\dots, \varepsilon\alpha^2(C), \varepsilon\alpha(C), C, \varepsilon\delta(C), \varepsilon\delta^2(C), \dots$ and our main result is

THEOREM 1. *Let $n_1, n_2, \dots, n_s$ be any sequence of integers. Then for each positive integer N , there exist a normal cascade C with rank $s - 1$ and tail $\geq N$, and a corresponding sequence ε of signs, such that*

$$n_i = \varepsilon\alpha^{s-i}(C) \quad \text{for } i = 1, 2, \dots, s.$$

CONJECTURE. We believe that some normal C and ε give also $n_i = \varepsilon\delta^{s-i}(C)$.

2. The proof. First of all, we wish to point out that the reader may find it helpful to display the Pascal's triangle as in Figure 1. If the binomial coefficients appearing in a normal cascade are marked in the figure, then the marks resemble water flowing from right to left:

$$\begin{array}{ccc}
 \cdots & \binom{2}{2} & \binom{1}{1} & \binom{0}{0} \\
 \cdots & \binom{3}{2} & \binom{2}{1} & \binom{1}{0} \\
 \cdots & \binom{4}{2} & \binom{3}{1} & \binom{2}{0} \\
 & \vdots & \vdots & \vdots
 \end{array}$$

FIGURE 1. Pascal's Triangle

A cascade C as given in (I) is called a *flat cascade* if $a_h - h = a_{h-1} - (h-1) = \cdots = a_t - t$. It has *rank* $a_h - h$. A *pattern* p_i for row $i \geq -1$ of length m is a sequence of signs $p_i = \langle \epsilon_1, \epsilon_2, \dots, \epsilon_m \rangle$ such that for any normal flat cascade C of rank i and length m , the number $p_i C$ is a constant. We call this constant the *value of* p_i , denoted by $v(p_i)$. Also, we shall denote the smallest $v(p_i) > 0$ when p_i runs through all patterns of row i by v_i and call it the *value of row* i . It is easy to observe that $v_0 = 1$, $v_1 = 1$ and $v_2 = 2$ with patterns $p_0 = \langle +1 \rangle$, $p_1 = \langle +1, -1 \rangle$ and $p_2 = \langle +1, -1, -1, +1 \rangle$. Also, for each pattern $p_i = \langle \epsilon_1, \dots, \epsilon_m \rangle$ for row i , the pattern $p_i - p_i = \langle \epsilon_1, \dots, \epsilon_m, -\epsilon_1, \dots, -\epsilon_m \rangle$ is a zero value pattern for row i . In fact, every sequence of signs is a pattern for row 0, because every binomial coefficient is 1.

LEMMA 1. For $i \geq 0$ and each pattern p for row i and every proper flat cascade C of rank i , we have $pC = p\mu(C) + p\rho(C)$.

PROOF. This follows directly from the relation

$$\binom{r}{s} = \binom{r-1}{s} + \binom{r-1}{s-1} \quad \text{for } r \geq s \geq 0 \text{ not } r = s = 0.$$

LEMMA 2. For $i \geq 0$ the pattern p is a zero value pattern for row $i-1$ if and only if p is a pattern for row i .

PROOF. If p is a zero value pattern for row $i-1$, then for any normal flat cascade C of rank i , we have, by Lemma 1, that $pC = p\mu(C) + p\rho(C) = p\rho(C)$. Now $\mu(C)$ is a flat cascade of rank $i-1$ which is normal except in the one trivial case $i=1$ and tail = 0 so $p\mu(C) = 0$. Therefore p is a pattern for row i .

Conversely, if p is a pattern for row i , then for any flat cascade C of rank $i-1$ and tail ≥ 1 , we have $pD = p\mu(D) + p\rho(D)$ where D is the flat cascade of rank i with $\mu(D) = C$. But then, $pC = p\mu(D) = 0$ since $p(D) = p\rho(D)$. Repetition shows $pB = 0$ for any flat cascade B of rank $i-2$ and tail ≥ 2 . Eventually we find that p is a zero value pattern for row 0. Then by the first part of the proof we know that p is a pattern for row 1 and so must be a zero value pattern. Continuing we find that p is a zero value pattern for row $i-1$ as required.

COROLLARY. There exists a pattern for row i for each $i = 0, 1, 2, \dots$.

LEMMA 3. If p_1, p_2 are patterns for row i with values u_1 and u_2 respectively, then there exists a pattern p for row i with value $g = \text{g.c.d.}(u_1, u_2)$.

PROOF. We can write $g = \lambda_1 u_1 + \lambda_2 u_2$ for suitable integers λ_1, λ_2 . Then all we need to do is take a pattern q consisting of $|\lambda_1|$ copies of p_1 and $|\lambda_2|$ copies of p_2 with appropriate signs.

As pointed out before, we have $v_0 = 1, v_1 = 1, v_2 = 2$, arising from the patterns $p_0 = \langle +1 \rangle, p_1 = \langle +1, -1 \rangle$ and $p_2 = \langle +1, -1, -1, +1 \rangle$ respectively. Let us try to estimate the value v_3 . Let C be a flat cascade of length 4, rank 3 and tail sufficiently large. Then we have $p_2 C - p_2 \rho(C) = p_2 \mu(C)$ by Lemma 1. But since $\mu(C)$ is a cascade of rank 2, we have $p_2 \mu(C) = v_2$, which implies that

$$p_2 C - p_2 \rho(C) = v_2.$$

Similarly,

$$p_2 \rho(C) - p_2 \rho^2(C) = v_2, \quad p_2 \rho^2(C) - p_2 \rho^3(C) = v_2$$

and

$$p_2 \rho^3(C) - p_2 \rho^4(C) = v_2.$$

Therefore

$$p_2 C - p_2 \rho^4(C) = 4v_2,$$

i.e.

$$(p_2 - p_2)(C + \rho^4(C)) = 4v_2 = 8,$$

which implies that $p_2 - p_2$ is a pattern for row 3 with value 8. This argument can easily be generalized to obtain

LEMMA 4. There exists a pattern for row i with value $2^{(i)}$ for $i = 0, 1, 2, \dots$

COROLLARY. The value v_i of row i divides $2^{(i)}$.

PROOF. This follows immediately from Lemmas 3 and 4.

We shall now prove Theorem 1 by induction on s . For $s = 1$, we need only choose a flat cascade of rank 0 say

$$C = \binom{a}{a} + \binom{a-1}{a-1} + \dots + \binom{a-|n_s|+1}{a-|n_s|+1} \quad \text{with } a - |n_s| + 1 \geq N.$$

The corresponding sequence of signs is either all positive or all negative depending on whether n_s is positive or not. Assume that the theorem holds for any sequence of $s-1$ integers and consider the sequence $\langle n_1, n_2, \dots, n_s \rangle$. Applying the induction hypothesis to $n_1, \dots, n_{s-1}$, there is a proper cascade

$$D = \binom{a_h}{h} + \cdots + \binom{a_t}{t}$$

with rank $s - 2$ and tail $\geq N + v_{s-1}$, and also a sequence $\epsilon^* = \langle \epsilon_h, \dots, \epsilon_t \rangle$ of signs such that $\epsilon^* D = n_{s-1}$, $\epsilon^* \alpha(D) = n_{s-2}$, $\dots$, $\epsilon^* \alpha^{s-2}(D) = n_1$. Let $C_1 = \delta D$ and $m = n_s - \epsilon^* C_1$. Without loss of generality, we may assume that $m \geq 0$. Let $[m/v] = k$ where $v = v_{s-1}$ and let $p = \langle p_1, \dots, p_l \rangle$ be a pattern for row $s - 1$ giving the value v . Now, let C_2 be the flat cascade of rank $s - 1$ with tail h and length kl and let C_3 be the flat cascade of rank 0 with head $(t - 2)$ and length $m - kv$. Finally in the obvious way put

$$\epsilon = +p \cdots + p + \epsilon^* + 1 \cdots + 1$$

and $C = C_2 + C_1 + C_3$. Then C is evidently a cascade of rank $s - 1$ with $\epsilon C = n_s$. Also by Lemma 2, $p\alpha^j C_2 = 0$ for $j = 1, 2, \dots$. Thus $\epsilon\alpha(C) = n_{s-1}$, $\epsilon\alpha^2(C) = n_{s-2}$, $\dots$, $\epsilon\alpha^{s-1}(C) = n_1$ completing the proof.

3. Remarks. (a) The proof of Theorem 1 in fact provides us with an algorithm to construct C and the corresponding sequence of signs. It is sufficient for the flat cascade of rank 0 with which we start to have a tail greater than

$$2^{\binom{1}{2}} + 2^{\binom{2}{2}} + \cdots + 2^{\binom{s-1}{2}}.$$

(b) The cascade of Theorem 1 can be chosen in infinitely many different ways.

(c) We omit the proof of

THEOREM 2. *Let $n_1, n_2, \dots, n_s$ be any sequence of integers. Then for each positive integer N there exists a flat cascade C with rank 0 and tail $\geq N$, and a corresponding sequence ϵ of signs such that*

$$|n_i - \epsilon \delta^{s-i}(C)| \leq 2^{\binom{i}{2}-1} \quad \text{for } i = 1, 2, \dots, s.$$

REFERENCES

1. J. B. Kruskal, *The number of simplices in a complex*, Mathematical Optimization Techniques, Univ. of California Press, Berkeley, 1963, pp. 251–278. MR 27 #4771.
2. G. Katona, *A theorem of finite sets*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 187–207. MR 45 #76.
3. D. E. Daykin, Jean Godfrey and A. J. W. Hilton, *Existence theorems for Sperner families*, J. Combinatorial Theory Ser. A 17 (1974), 245–251.
4. D. E. Daykin, *A simple proof of the Kruskal-Katona theorem*, J. Combinatorial Theory Ser. A 17 (1974), 252–253.
5. ———, *Cascade algorithms giving Katona-type inequalities*, Nanta. Math. (to appear).
6. ———, *The average size set in an antichain*, Nanta. Math. (to appear).

DEPARTMENT OF MATHEMATICS, NANYANG UNIVERSITY, SINGAPORE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF READING, READING, ENGLAND