ON TOPOLOGICAL METHODS IN HOMOLOGICAL ALGEBRA

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ABSTRACT. We give an appropriate extension of the concept of "tower of surjections" to arbitrary inverse systems. We introduce a natural closed model structure (in the sense of D. Quillen) on the category of pro-(Simplicial Abelian Groups) and interpret our condition as the definition of fibrant object.

1. Introduction. Consider the following well-known proposition.

(1.1) **PROPOSITION.** *Let*

$$G_0 \leftarrow \cdots \leftarrow G_i \leftarrow G_{i+1} \leftarrow \cdots$$

be a tower of surjections of abelian groups. Then

- (a) *the natural maps $\lim \{G_i\} \rightarrow G_{i_0}$ are surjections, and*
- (b) *$\lim^s \{G_i\} = 0$ for $s > 0$.*

It is also well known (see, for example, J. Keesling [11]) that (a) fails in general for uncountable inverse systems bonded by surjections. We shall show in §2 that (b) also fails in general for such systems. In §3 we shall show that (a) and (b) both hold for inverse systems $\{G_i\}$ indexed by *cofinite strongly directed sets* with the property

(1.2) the induced maps $G_{i_0} \rightarrow \lim_{i < i_0} \{G_i\}$ are surjections.

A directed set $I = \{i\}$ called *strongly directed* if $i \leq i'$ and $i' \leq i$ imply $i = i'$. I is called *cofinite* if each element $i \in I$ has finitely many predecessors. In [6], [7], we extend a result of S. Mardešić [12] to show that every filtering category J (a notion more general than directed set) is equivalent to a cofinite strongly directed set.

In §4 we shall interpret condition (1.2) as the definition of *fibrant* object in a suitable closed model structure (D. Quillen [14]) on pro-(Simplicial Abelian Groups) [6], [7], [8], and indicate several applications.

Z. Z. Yeh [17], J.-E. Roos [15], and J.-L. Verdier [16] have shown that part (b) of Proposition (1.1) holds for *flasque* inverse systems; C. U. Jensen [10]

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extended their results to *weakly flasque* systems. (A subset J of a directed set I is called *ordered* if $i \in J$ and $i' < i$ imply $i' \in J$. An inverse system $\{G_i\}$ of abelian groups indexed by a directed set I is called *flasque* (resp., *weakly flasque*) if for each ordered (resp., directed) subset J of I the induced map $\lim_{i \in J} \{G_i\} \rightarrow \lim_{i \in J} \{G_i\}$ is surjective.) In §3 we shall show that flasque is equivalent to condition (1.2).

(1.3) *Convention.* In the rest of this paper, we shall assume that all inverse systems are indexed by cofinite strongly directed sets.

A. Grothendieck [8] associated to a category C the category $\text{pro-}C$ of inverse systems over C ; see M. Artin and B. Mazur [1, Appendix] for a description of $\text{pro-}C$.

2. Inadequacy of the Mittag-Leffler condition. The inverse limit, $\lim$, and its right derived functors $\lim^s$ are functors from $\text{pro-}(\text{Abelian Groups})$ to Abelian Groups ([17], [4], see [6] for a proof).

(2.1) **DEFINITIONS** (see, e.g., [3]). (a) An inverse system $\{G_i\}$ is called *movable* if for each i there exists a $j > i$ such that for any $k > j$ there exists a filler in the diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\quad} & G_i \\ & \nwarrow \text{dashed} & \nearrow \\ & G_j & \end{array}$$

(b) $\{G_i\}$ is called *Mittag-Leffler* (M-L) if $\{G_i\}$ is isomorphic in $\text{pro-}(\text{Abelian Groups})$ to an inverse system $\{G'_j\}$ whose bonding maps are surjections.

(2.2) **LEMMA** (see, e.g., [3]). *A movable inverse system is Mittag-Leffler.*

We omit the routine proof.

(2.3) **PROPOSITION.** *There is a short exact sequence of Mittag-Leffler pro-abelian groups on which $\lim$ is not exact; hence $\lim^1$ does not vanish on all Mittag-Leffler pro-abelian groups.*

PROOF. In [10], Keesling constructs a movable pair $\{(X_i, A_i)\}$ in the category $\text{pro-}(\text{ANR Pairs})$, hence a *movable system* of long exact sequences

$$(2.4) \quad \left\{ \dots \rightarrow H_1(X_i) \rightarrow H_1(X_i, A_i) \xrightarrow{\partial_i} H_0(A_i) \rightarrow \dots \right\},$$

such that the sequence

$$(2.5) \quad \lim \{H_1(X_i)\} \rightarrow \lim \{H_1(X_i, A_i)\} \rightarrow \lim \{H_0(A_i)\}$$

is not exact at the middle term.

It is easy to show that the kernels and images in sequence (2.4) are movable, hence M-L. If $\lim$ were exact on such systems, we would have

$$\begin{aligned} \operatorname{Im}(\lim\{H_1(X_i)\} \rightarrow \lim\{H_1(X_i, A_i)\}) &= \lim\{\operatorname{Im}(H_1(X_i) \rightarrow H_1(A_i))\} \\ &= \ker(\lim\{H_1(X_i, A_i)\} \rightarrow \lim\{H_0(A_i)\}), \end{aligned}$$

in sequence (2.5), contradicting Keesling. Hence $\lim$ is not exact on M-L pro-abelian groups. $\square$

3. The strong Mittag-Leffler condition.

(3.1) DEFINITION. An inverse system of abelian groups, $\{G_i\}$, indexed by a cofinite directed set, is said to satisfy the *strong Mittag-Leffler* (S-M-L) condition if, for each i_0 , the induced map $G_{i_0} \rightarrow \lim_{i < i_0} \{G_i\}$ is surjective. More generally, any pro-abelian group isomorphic to an S-M-L pro-abelian group in pro-(Abelian Groups) will be called S-M-L.

(3.2) REMARKS. Clearly S-M-L implies M-L, and the two conditions are equivalent on towers (inverse systems indexed by the natural numbers).

(3.3) PROPOSITION. *An inverse system $\{G_i\}$ is strongly Mittag-Leffler if and only if $\{G_i\}$ is isomorphic in pro-(Abelian Groups) to a flasque inverse system.*

PROOF. Clearly, if $\{G_i\}$ is flasque, the natural maps $G_{i_0} \rightarrow \lim_{i < i_0} \{G_i\}$ are surjections.

Conversely, assume that for each i_0 the natural maps $G_{i_0} \rightarrow \lim_{i < i_0} \{G_i\}$ are surjections. Let I denote $\{i\}$, and let J be an ordered subset of I . Given an element $g_J = \{g_i | i \in J\}$ of $G_J = \lim_{i \in J} \{G_i\}$ we shall find an element $g_I = \{g_i | i \in I\}$ which extends g_J ; this will show that $\{G_i\}$ is flasque, as required.

We shall define the required elements g_i in $I \setminus J$ by induction. Suppose that for a given i_0 , and for all $i < i_0$, elements g_i in G_i have been chosen so that for $i' < i$, g_i maps to $g_{i'}$ in $G_{i'}$. (If i_0 has no predecessors in $I \setminus J$, this assumption is clearly satisfied.) By hypothesis we may define g_{i_0} in G_{i_0} such that, for all $i < i_0$, g_{i_0} maps to g_i in G_i . Because I is a cofinite *strongly* directed set, this process yields the required element g_I in $\lim_{i \in I} \{G_i\}$. $\square$

4. Relation with pro-homotopy. We shall give a homotopy-theoretic interpretation of the S-M-L condition. D. Quillen [14, §1.1] formalized axiomatic homotopy theory by introducing the concept of *closed model category*. A closed model category C consists of an underlying category C together with three classes of maps; cofibrations, fibrations, and equivalences. These maps are required to satisfy the "usual properties." In particular, given a solid-arrow commutative diagram of the form

$$(4.1) \quad \begin{array}{ccc} A & \xrightarrow{\quad} & Y \\ i \downarrow & \nearrow f & \downarrow p \\ X & \xrightarrow{\quad} & B \end{array}$$

in which i is a cofibration, p is a fibration, and i or p is an equivalence, there

exists a filler f . An object X is called *fibrant* if the natural map $X \rightarrow *$ is a fibration.

The category SS of simplicial sets (see, e.g., J. P. May [13]) is the prototype closed model category [14, §II.3]; the cofibrations are the inclusions, the fibrations are the Kan fibrations, and the equivalences are the weak homotopy equivalences. The category $S(\text{Abelian Groups})$ of simplicial abelian groups inherits a closed model structure from SS [14, §II.3]. *Caution.* Not all inclusions are cofibrations in $S(\text{Abelian Groups})$. In the following discussion let C be either SS or $S(\text{Abelian Groups})$. The case $C = SS$ was first developed in [9].

(4.2) DEFINITIONS [7]. A level map $\{f_i\}$ in $\text{pro-}C$ is a *cofibration* (resp., *equivalence*) if each f_i is a cofibration (resp., equivalence). More generally, any composite of retracts of such maps is a cofibration (resp., equivalence). Fibrations are defined by lifting property (4.1).

(4.3) THEOREM [7]. *pro-}C with the above structure is a closed model category. An object }X of pro-}C is fibrant if and only if }X is a retract of an object }{Y_i indexed by a cofinite strongly directed set }{i such that, for each }i_0, the induced map }Y_{i_0} \rightarrow \lim_{i < i_0} \{Y_i\} is a fibration.*

We relate Theorem (4.3) to the homological algebra of §3 as follows. Quillen [14, §II.3] showed that a map $f: G \rightarrow H$ of abelian groups is a surjection if and only if f is a fibration in $S(\text{Abelian Groups})$. Hence a pro-abelian group X is S-M-L if and only if X is fibrant in $\text{pro-}S(\text{Abelian Groups})$. This allows us to give another proof of the following result of Yeh [17] and Roos [15]; see also Verdier [16] and Jensen [10].

(4.4) THEOREM [6], [7] (compare Proposition (1.1)(b) above). *For }s > 0, \lim^s vanishes on S-M-L pro-abelian groups.*

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