ON THE LU QI-KENG CONJECTURE

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ABSTRACT. We shall give a complete answer to the Lu Qi-keng conjecture for finite Riemann surfaces. Our result is that every finite Riemann surface which is not simply-connected is never a Lu Qi-keng domain, i.e. the Bergman kernel $K(z, t)$ of it has zeros for suitable t 's.

1. Let Ω be a Riemann surface and let Γ be the Hilbert space of analytic differentials with finite norm on Ω . Unless Ω is conformally equivalent to the Riemann sphere less a compact set of capacity zero, Γ contains nontrivial differentials and the Bergman kernel of Ω , denoted by $K(z, \bar{t}) dz d\bar{t}$ is not identically equal to zero. It is characterized by the reproducing property

$$\alpha(z) = \frac{1}{2i} \iint_{\Omega} K(z, \bar{t}) d\bar{t} \wedge \alpha(t) dt, \quad \alpha(t) dt \in \Gamma.$$

Suppose that Ω has the Green's function $g(z, t)$. Then $K(z, \bar{t})$ is given by

$$K(z, \bar{t}) = -\frac{2}{\pi} \frac{\partial^2 g(z, t)}{\partial z \partial \bar{t}}.$$

This identity is due to Schiffer [3].

The following conjecture is called the Lu Qi-keng conjecture for a domain $\Omega \subset C^n$ by Skwarczynski [4]: The Bergman kernel $K(z, \bar{t})$ of Ω has no zeros in $\Omega \times \Omega$. This conjecture relates to a regularity problem of the invariant metric [1], [4]. He proved that some annulus ($\subset C^1$) is not a Lu Qi-keng domain [4] and later Rosenthal showed that nor is every annulus [2].

2. We prove

THEOREM. *Let Ω be a finite open Riemann surface which is not simply-connected. Then the Bergman kernel $K(z, \bar{t})$ of Ω has exactly $n + 2p - 1$ zeros for suitable t 's. Here n and p denote the numbers of the boundary contours and the genus of Ω , respectively.*

PROOF. Let $L(z, t) dz dt$ be the adjoint kernel of $K(z, \bar{t}) dz d\bar{t}$ given by

$$L(z, t) = -\frac{2}{\pi} \frac{\partial^2 g(z, t)}{\partial z \partial t}$$

Received by the editors January 5, 1976.

AMS (MOS) subject classifications (1970). Primary 30A31.

Key words and phrases. Kernel function, Bergman kernel, Riemann surface.

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which is a meromorphic differential on Ω with respect to z and t [3]. We denote by $\hat{\Omega}$ the double of Ω . Then the differentials $K(z, \bar{t})dz$ and $L(z, t)dz$ with fixed $t \in \Omega$ are meromorphically extended to $\hat{\Omega}$ by the relations:

$$\alpha(z)dz = \begin{cases} K(z, \bar{t})dz & \text{for } z \in \bar{\Omega}, \\ -L(z^*, t)dz & \text{for } z \in \hat{\Omega} - \Omega, \end{cases}$$

$$\beta(z)dz = \begin{cases} L(z, t)dz & \text{for } z \in \bar{\Omega}, \\ -K(t, \bar{z}^*)dz & \text{for } z \in \hat{\Omega} - \Omega. \end{cases}$$

Note that z and t are used as local parameters and z^* denotes the image of z under the canonical involutive map of $\hat{\Omega}$.

It is well known that the quadratic differential $\alpha\beta dz^2$ is a positive differential on Ω . Since $L(z, t)dz$ has one double pole at t , $\alpha\beta dz^2$ has $4p + 2n - 2$ zeros on $\bar{\Omega}$, multiplicities being counted by half on $\partial\Omega$. It is easily verified that the t can be chosen at a point on $\partial\Omega$ and that the properties mentioned above still hold. We take a local parameter about $t \in \partial\Omega$ on a half neighborhood so that the boundary on $\partial\Omega$ and t correspond to a segment on the real axis and to the origin respectively. Then for $t = x + iy$,

$$\left. \frac{\partial g(z, t)}{\partial \bar{t}} \right|_{t=0} = \frac{i}{2} \left. \frac{\partial g(z, t)}{\partial y} \right|_{t=0},$$

$$\left. \frac{\partial g(z, t)}{\partial t} \right|_{t=0} = -\frac{i}{2} \left. \frac{\partial g(z, t)}{\partial y} \right|_{t=0}.$$

Hence we have

$$K(z, \bar{t})dz = -L(z, t)dz, \quad z \in \Omega, t \in \partial\Omega.$$

On the other hand,

$$H(z) = \partial g(z, t)/\partial y|_{t=0}$$

is a positive harmonic function on Ω and vanishes identically on $\partial\Omega$ except for $z = t$. Since $H(z)$ can be harmonically extended to $\hat{\Omega}$ except for $z = t$, the meromorphic differential dH has $2p + n - 1$ zeros in Ω . Note that $H(z)$ has no critical points on $\partial\Omega$ and that dH has one double pole at t . This means that $K(z, \bar{t})$ and $L(z, t)$ both have $2p + n - 1$ zeros for $t \in \partial\Omega$.

We let the reference point τ of $K(z, \bar{\tau})dz$ tend to $t \in \partial\Omega$. Then $\{K(z, \bar{\tau})\}$ tends to $K(z, \bar{t})$ uniformly on every compact set in Ω . Thus by Hurwitz's theorem the zeros of $K(z, \bar{\tau})$ cluster on the zeros of $K(z, \bar{t})$. This completes the proof.

3. If Ω is a compact Riemann surface, the Bergman kernel $K(z, \bar{t})dz$ is an analytic differential on Ω . In this case, $K(z, \bar{t})$ has exactly $2(p - 1)$ zeros if the genus p of Ω is greater than 1.

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