OPEN SUBSETS OF R^∞ ARE STABLE

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ABSTRACT. Let U be an open subset of $R^\infty = \text{dir lim } R^n$, where R denotes the reals. We show that $U \times R^\infty$ is homeomorphic to U . Combined with previous work of the author we obtain the corollary that two open subsets of R^∞ are homeomorphic if and only if they have the same homotopy type.

Let R denote the reals, and let $R^\infty = \text{dir lim } R^n$. Here we prove the following.

THEOREM 1. *If U is an open subset of R^∞ then $U \times R^\infty$ is homeomorphic to U .*

In [2, Theorem II-8] it is shown that if two paracompact, connected R^∞ -manifolds N , M have the same homotopy type then $N \times R^\infty$ and $M \times R^\infty$ are homeomorphic. Since open subsets of R^∞ are paracompact [2, Proposition III-1] we obtain the following.

COROLLARY 2. *Two open subsets of R^∞ are homeomorphic iff they have the same homotopy type.*

PROOF OF THEOREM 1. We regard R^∞ as $\bigcup \{R^n | n = 1, 2, \dots\}$, and we identify R^n with $R^n \times \{0\} \subset R^{n+k}$, $k \geq 1$, so that $R^n \subset R^{n+k} \subset R^\infty$. Let U be an open subset of R^∞ . We may assume that U is connected. By [2, Proposition III-2] (or by elementary reasoning) $U = \bigcup \{C_n | n = 1, 2, \dots\}$ where C_n is compact, $C_n \subset C_{n+1}$, $n \geq 1$, and where a subset V of U is open in U iff $V \cap C_n$ is open in C_n , $n \geq 1$. An elementary argument shows we may assume additionally that $C_n \subset R^n$.

In what follows the word manifold will be used only for a compact, p.l. (piecewise linear) manifold, possibly with boundary. We observe that if K is any compact set and if $K \subset W$ where W is open in R^n , then there is an n -manifold M such that $K \subset M \subset W$. This manifold may be obtained, for example, by first finding a polyhedron in W containing K and then taking a regular neighborhood of this polyhedron in W . For $n \geq 1$ and $\epsilon > 0$ let

$$D(n, \epsilon) = \{x = (x_1, \dots, x_n) \in R^n | |x_i| \leq \epsilon, i = 1, \dots, n \text{ and } x_n \geq 0\}.$$

For $n \geq 1$ let $U^n = U \cap R^n$.

Received by the editors October 27, 1975 and, in revised form, February 23, 1976.

AMS (MOS) subject classifications (1970). Primary 57A20, 54B99.

Key words and phrases. Open subset, direct limit, stability, homotopy type.

¹Supported in part by NSF Grant GP-43772.

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To begin, choose a 2-manifold M_1 such that $C_2 \subset M_1 \subset U^2$. Choose $\delta_1 > 0$ such that $M_1 \times [0, \delta_1] \subset U^3$. Let $N_1 = \partial(M_1 \times [0, \delta_1])$, where ∂ denotes manifold boundary. Choose $\varepsilon_1 > 0$ such that $N_1 \times D(1, \varepsilon_1) \subset U^4$. Now choose a 4-manifold M_2 such that $[N_1 \times D(1, \varepsilon_1)] \cup C_4 \subset M_2 \subset U^4$. Choose $\delta_2 > 0$ such that $M_2 \times [0, \delta_2] \subset U^5$, and let $N_2 = \partial(M_2 \times [0, \delta_2])$. Choose $\varepsilon_2 > 0$ such that $N_2 \times D(3, \varepsilon_2) \subset U^8$. Inductively, suppose we have chosen $N_1, \dots, N_k$ and $\varepsilon_1, \dots, \varepsilon_k$ where N_i is a closed 2^i -manifold and $\varepsilon_i > 0$, $1 \leq i \leq k$, where

$$[N_{i-1} \times D(2^{(i-1)} - 1, \varepsilon_i)] \cup C_{2^i} \subset N_i \subset U^{2^{i+1}},$$

$2 \leq i \leq k$, and where $N_k \times D(2^k - 1, \varepsilon_k) \subset U^{2^{(k+1)}}$. Choose a $2^{(k+1)}$ -manifold M_{k+1} such that

$$[N_k \times D(2^k - 1, \varepsilon_k)] \cup C_{2^{(k+1)}} \subset M_{k+1} \subset U^{2^{(k+1)}}.$$

Choose $\delta_{k+1} > 0$ such that $M_{k+1} \times [0, \delta_{k+1}] \subset U^{2^{(k+1)+1}}$, and let $N_{k+1} = \partial(M_{k+1} \times [0, \delta_{k+1}])$. Choose $\varepsilon_{k+1} > 0$ such that

$$N_{k+1} \times D(2^{(k+1)} - 1, \varepsilon_{k+1}) \subset U^{2^{(k+2)}}.$$

In this way we obtain $N_k, \varepsilon_k > 0$, $k > 1$, such that N_k is a closed 2^k -manifold and

$$[N_k \times D(2^k - 1, \varepsilon_k)] \cup C_{2^k} \subset N_{k+1} \subset U^{2^{(k+1)+1}}.$$

For $k \geq 1$ let $E_k = \partial D(2^k - 1, \varepsilon_k)$ and let $F_k = \partial D(2^k - 1, 2^k - 1)$. Then $N_k \times E_k$ and $N_k \times F_k$ are closed $(2^{(k+1)} - 2)$ -manifolds and $N_k \times E_k \subset N_{k+1}$. For $k \geq 1$ define a linear homeomorphism $h_k: N_k \times E_k \rightarrow N_k \times F_k$ by $h_k(x, y) = (x, [(2^k - 1)/\varepsilon_k]y)$ where $x \in N_k$ and $y \in E_k$. Consider the diagram

$$\begin{array}{ccccccc} N_1 \times E_1 & \xrightarrow{\alpha_1} & \cdots & N_k \times E_k & \xrightarrow{\alpha_k} & N_{k+1} \times E_{k+1} & \xrightarrow{\alpha_{k+1}} \cdots \\ \downarrow h_1 & & & \downarrow h_k & & \downarrow h_{k+1} & \\ N_1 \times F_1 & \xrightarrow{\beta_1} & \cdots & N_k \times F_k & \xrightarrow{\beta_k} & N_{k+1} \times F_{k+1} & \xrightarrow{\beta_{k+1}} \cdots \end{array}$$

Here α_k is the inclusion $\alpha_k(x, y) = ((x, y), 0)$ where $x \in N_k$, $y \in E_k$, $(x, y) \in N_{k+1}$, $0 \in E_{k+1}$, and β_k is the inclusion $\beta_k(x, y) = ((x, 0), (y, 0))$ where $x \in N_k$, $y \in F_k$, $(x, 0) \in N_{k+1}$ and $(y, 0) \in F_{k+1}$. Since

$$N_{k+1} \supset N_k \times D(2^k - 1, \varepsilon_k)$$

and since E_k contracts in $D(2^k - 1, \varepsilon_k)$, the p.l. embedding $h_{k+1}\alpha_k$ is easily seen to be homotopic to the map $(x, y) \rightarrow ((x, 0), 0)$. Since

$$F_{k+1} = \partial D(2^{k+1} - 1, 2^{k+1} - 1) \supset D(2^k - 1, 2^k - 1)$$

and since F_k contracts in $D(2^k - 1, 2^k - 1)$, the p.l. embedding $\beta_k h_k$ is easily seen to be homotopic to the map $(x, y) \rightarrow ((x, 0), (0, 0))$. Thus, $h_{k+1}\alpha_k$ and $\beta_k h_k$ are homotopic, $k \geq 1$.

We proceed to inductively replace the p.l. homeomorphisms h_k by p.l. homeomorphisms g_k such that $g_{k+1}\alpha_k = \beta_k g_k$. Let $g_1 = h_1$. Now $N_1 \times E_1$ is a closed 2-manifold and $N_2 \times F_2$ is a closed 6 = (2(2) + 2)-manifold. By [3, Corollary 5.9, p. 65] $h_2\alpha_1$ and $\beta_1 g_1$ are ambient isotopic. Thus, there is a level preserving p.l. homeomorphism $H_2: N_2 \times F_2 \times I \rightarrow N_2 \times F_2 \times I$ such that $(H_2)_0 = \text{id}$ and $(H_2)_1 h_2\alpha_1 = \beta_1 g_1$. Let $g_2 = (H_2)_1 h_2$. Then g_2 is a p.l. homeomorphism and $g_2\alpha_1 = \beta_1 g_1$. Also, if $\pi: N_2 \times F_2 \times I \rightarrow N_2 \times F_2$ is the projection map, then $\pi \circ H \circ (h_2 \times \text{id}): N_2 \times F_2 \times I \rightarrow N_2 \times F_2$ is a homotopy between h_2 and g_2 . Thus, $\beta_2 g_2 \sim \beta_2 h_2 \sim h_3\alpha_2$. (By " $\sim$ " we denote "is homotopic to".) Suppose, inductively, that we have defined $g_1, \dots, g_k$ such that $g_i\alpha_{i-1} = \beta_{i-1} g_{i-1}$, $i = 2, \dots, k$, and $\beta_k g_k \sim h_{k+1}\alpha_k$. Noting that $\dim(N_{k+1} \times F_{k+1}) = 2^{(k+2)} - 2 = 2(2^{k+1} - 2) + 2 = 2 \dim(N_k \times E_k) + 2$ and again applying [3, Corollary 5.9, p. 65] we know that $\beta_k g_k$ and $h_{k+1}\alpha_k$ are ambient isotopic. Proceeding as in the definition of g_2 we obtain a p.l. homeomorphism $g_{k+1}: N_{k+1} \times E_{k+1} \rightarrow N_{k+1} \times F_{k+1}$ such that $g_{k+1}\alpha_k = \beta_k g_k$ and $\beta_{k+1} g_{k+1} \sim h_{k+2}\alpha_{k+1}$. By induction we obtain our desired sequence of p.l. homeomorphisms $\{g_n\}$.

The sequence $\{g_n\}$ induces a homeomorphism

$$g_\infty: \text{dir lim}\{N_k \times E_k; \alpha_k\} \rightarrow \text{dir lim}\{N_k \times F_k; \beta_k\}.$$

Our theorem will be established if we show that $\text{dir lim}\{N_k \times E_k; \alpha_k\} = U$ and $\text{dir lim}\{N_k \times F_k; \beta_k\} = U \times R^\infty$. Clearly these equalities hold for the underlying point sets. Let $\mathcal{O} \subset U$ be such that $\mathcal{O} \cap (N_k \times E_k)$ is open in $N_k \times E_k$, $k \geq 1$. Given any C_n we have $C_n \subset N_j$ for any j such that $2^j \geq n$. Since $\mathcal{O} \cap (N_j \times E_j)$ is open in $N_j \times E_j$ we have $\mathcal{O} \cap C_n = [\mathcal{O} \cap (N_j \times E_j)] \cap C_n$ is open in $(N_j \times E_j) \cap C_n = C_n$. By choice of $\{C_n\}$, $\mathcal{O}$ is open in U . On the other hand, if $\mathcal{O}$ is open in U then, since each $N_k \times E_k \subset U^{2^k+1}$ has the relative topology induced from U , $\mathcal{O} \cap (N_k \times E_k)$ is open in $N_k \times E_k$, $k \geq 1$. Thus, $U = \text{dir lim}\{N_k \times E_k; \alpha_k\}$. For the second direct limit note that $U \times R^\infty$, being homeomorphic to an open subset of R^∞ (since $R^\infty \times R^\infty \cong R^\infty$ by [1, Corollary III-1]), is a connected paracompact R -manifold. By [2, Proposition III-2] $U \times R^\infty = \bigcup \{K_n | n = 1, \dots, \infty\}$ where each K is compact, and where $V \subset U \times R^\infty$ is open iff $V \cap K_n$ is open in K_n , all $n \geq 1$. Let $V \subset U \times R^\infty$ be such that $V \cap (N_k \times F_k)$ is open in $N_k \times F_k$, all k . Given K_n we can choose j (e.g. see [1, Lemma III-6]) such that

$$\begin{aligned} K_n &\subset \pi_U(K_n) \times \pi_{R^\infty}(K_n) \subset C_{2^j} \times D(2^{(j-1)} - 2, 2^{(j-1)} - 2) \\ &\subset C_{2^j} \times \partial D(2^{(j-1)} - 1, 2^{(j-1)} - 1) \subset N_j \times F_j. \end{aligned}$$

Since $V \cap (N_j \times F_j)$ is open in $N_j \times F_j$ we then have $V \cap K_n = [V \cap (N_j \times F_j)] \cap K_n$ is open in $(N_j \times F_j) \cap K_n = K_n$. Thus, V is open in $U \times R^\infty$. Conversely, if V is open in $U \times R^\infty$ then, since each $N_k \times F_k$ has the relative topology induced from $U \times R^\infty$, $V \cap (N_k \times F_k)$ is open in $N_k \times F_k$, $k \geq 1$. It follows that $U \times R^\infty = \text{dir lim}\{N_k \times F_k; \beta_k\}$, and our proof is complete.

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