SHORTER NOTES

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THE QUADRATIC PUTNAM-FUGLEDE THEOREM

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ABSTRACT. We give a new sufficient condition for an invariant subspace for a normal operator to be reducing. This result constitutes a generalization of the Putnam-Fuglede Theorem to quadratic operator equations.

Let $\mathbf{H}$ be a complex Hilbert space and N a normal operator on $\mathbf{H}$.

THEOREM 1. *Suppose $\mathbf{M}$ is a reducing subspace for N and $\mathbf{N}$ is an invariant subspace for N such that the projection of $\mathbf{N}$ on $\mathbf{M}$ has a continuous inverse. Then $\mathbf{N}$ reduces N .*

Phrasing Theorem 1 in terms of operator equations yields a quadratic version of the Putnam-Fuglede Theorem [1, p. 20] or [2]:

THEOREM 2. *Suppose $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a normal operator on the direct sum of two Hilbert spaces, $\mathbf{H} \oplus \mathbf{K}$. Let K and G be two bounded linear transformations from $\mathbf{H}$ to $\mathbf{K}$ satisfying the equation*

$$(1) \quad XBX + XA - DX - C = 0$$

such that $I + K^*G$ is invertible. Then K satisfies

$$(2) \quad XC^*X + XA^* - D^*X - B^* = 0$$

if and only if G does.

REMARK. To obtain the Putnam-Fuglede Theorem from Theorem 2, take $B = C = 0 = K$.

PROOF OF THEOREM 1. Let

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

with respect to the decomposition $\mathbf{H} = \mathbf{M} \oplus \mathbf{M}^\perp$. The components are normal. The hypothesis on $\mathbf{N}$ implies that it is the graph of a bounded linear transformation K from $\mathbf{M}$ to $\mathbf{M}^\perp$. The invariance of $\mathbf{N}$ is equivalent to the

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equation $KN_1 = N_2K$. The Putnam-Fuglede Theorem applies even though the N_i act on different spaces. We obtain $KN_1^* = N_2^*K$, which says that the graph of K is invariant under N^* as desired.

PROOF OF THEOREM 2. The fact that K and G satisfy (1) says that $\mathcal{G}(K)$ and $\mathcal{G}(G)$ (the graphs of K and G) are invariant under N . The projection of $\mathbf{H} \oplus \mathbf{K}$ on the graph of K is given by

$$\begin{pmatrix} L & LK^* \\ KL & KLK^* \end{pmatrix},$$

where $L = (I + K^*K)^{-1}$. This projection satisfies the hypothesis of Theorem 1 with $\mathbf{N} = \mathcal{G}(G)$ if $I + K^*G$ is invertible. Equation (2) is the condition for the invariance of $\mathcal{G}(X)$ under N^* , so Theorem 2 follows from Theorem 1 and the symmetry of the hypothesis on K and G .

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