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THREE IDENTITIES BETWEEN STIRLING NUMBERS AND THE STABILIZING CHARACTER SEQUENCE

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ABSTRACT. Let χ denote the stabilizing character of the action of the finite group G on the finite set X. Let χ_k denote $|G|^{-1}\Sigma_{\sigma \in G}\chi(\sigma)^k$. It is well known that χ_k is the number of orbits of the induced action of G on the Cartesian product $X^{(k)}$. We show if G is a least $(k-1)$-fold transitive on X, then χ_k can be expressed in terms of Stirling numbers of both kinds. Three identities between Stirling numbers and the stabilizing character sequence are obtained.

Let X be a finite set and let G be a finite group of permutations acting on X. We adopt essentially the notation of Lang [5]. For $x \in X$, $G_x = \{\sigma \in G: \sigma(x) = x\}$ is the stabilizer subgroup of x, and $Gx = \{\sigma(x): \sigma \in G\}$ is the orbit of x. Further, if $A \subseteq X$, $G_A = \{\sigma \in G: \sigma(a) = a$ for all $a \in A\}$. Note $G_\emptyset = G$ and $\sigma G_A \sigma^{-1} = G_{\sigma A}$. It is well known ([5], [7] and [11], for example) that 'belonging to the same orbit' is an equivalence relation on X so that the G-orbits partition X. Also well known ([5], [7] and [11]) is $\sigma(x) = \tau(x)$ iff σ and τ lie in the same coset of G_x. Therefore, for all $x \in X$, $|Gx| = [G: G_x]$. (Here $|A|$ denotes the cardinality of the set A.) Summarizing these two remarks is the classical theorem:

THEOREM 1. $|X| = \Sigma[G: G_x]$, *where the summation is taken over one representative of each orbit.*

For $\sigma \in G$, $\chi(\sigma)$ denotes the number of elements of X fixed by σ. χ is called the stabilizing character of the action of G on X. The normalized sum $|G|^{-1}\Sigma_{\sigma \in G}\chi^k(\sigma)$ we denote as χ_k. Burnside [1] (see also Pólya [8]) proved

THEOREM 2. $\chi_1 = t_0$ *where* t_0 *denotes the number of G-orbits in X. For* $k \geqslant 2$, χ_k *is the number of orbits of the induced action of G on the Cartesian product* X^k.

Following Shapiro's hint [11], we note Theorem 2 can easily be shown by counting $\{(\sigma, \bar{x}): \sigma(\bar{x}) = \bar{x}, \bar{x} \in X^k\}$ in two different ways. Burnside [1] also proved

THEOREM 3. *If G is transitive on X, then* $\chi_2 = t_1$ *where* t_1 *denotes the number of* G_x*-orbits in X.*

In this paper we obtain an extension of Theorem 3 to normalized sums of

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higher powers of χ. In particular, under suitable transitivity restrictions, we obtain formulas relating Stirling numbers and the stabilizing character sequence. The main tool we use is an extension of the standard proof (see, for example, Hall [4, p. 280]) of Theorem 3. Thanks are due to Otto Ruehr and the referee.

First we state the basic relations between Stirling numbers and factorial polynomials. These relations are well known and are stated, for example, in Liu [6] and Riordan [9]. We choose the notation given by Gould [3].

The factorial polynomials are defined by $Y^{(0)} = 1$ and

$$Y^{(n)} = Y(Y-1)\dots(Y-n+1) \quad \text{for } n \geqslant 1.$$

Then Stirling numbers of the first kind are defined by

$$\sum_{i=0}^{n} S_1(n,i)Y^i = Y^{(n)} \qquad (n \geqslant 1)$$

and satisfy the recurrence relations

$$S_1(n,i) = S_1(n-i, i-1) - (n-1)S_1(n-i,i) \qquad (1 \leqslant i \leqslant n)$$

where $S_1(n, 0) = 0$ for all n. Stirling numbers of the second kind are defined by

$$\sum_{i=0}^{n} S_2(n,i)Y^{(i)} = Y^n \qquad (n \geqslant 1)$$

and satisfy the recurrence relations

$$S_2(n,i) = iS_2(n-1,i) + S_2(n-1,i-1) \qquad (1 \leqslant i \leqslant n)$$

where $S_2(n, 0) = 0$ for all n. The product of the two $n \times n$ matrices $(S_1(i,j)) \cdot (S_2(i,j))$ is the $n \times n$ identity so that

$$\sum_{k=1}^{n} S_1(i,k)S_2(k,j) = \delta_{ij}, \text{ Kronecker delta.}$$

For our purposes it will be convenient to denote by $P_n(n_1, \dots, n_j)$ the product

$$P_n(n_1, \dots, n_j) = (-1)^j S_1(n, n_1) \cdot S_1(n_1, n_2) \dots S_1(n_{j-1}, n_j)$$

where $n > n_1 > \cdots > n_j \geqslant 1$ is a decreasing sequence of integers. We also denote by $\Sigma' P_n(n_1, \dots, n_j)$ the sum of $P_n(n_1, \dots, n_j)$ over all $j, n_1, \dots, n_j$ where $n > n_1 > \cdots > n_j \geqslant 1$ is a decreasing sequence of integers.

Recall that G is k-fold transitive on X in case for all k-element subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of X there exists $\sigma \in G$ with $\sigma(a_i) = b_i$ for $1 \leqslant i \leqslant k$. If G is $|X|$-fold transitive we will say that G is k-fold transitive for all positive integers k. We will require the following lemma, which is probably well known, but is given here for completeness.

LEMMA 4. *If G is k-fold transitive on X and A is any k-element subset of X, then*

$$|X|(|X|-1)\dots(|X|-k+1) = [G: G_A].$$

PROOF. Let $A = \{x_1, \dots, x_k\}$. Then for $0 \leqslant i < k$, $G_{x_1, \dots, x_i}$ is transitive on the set $X \setminus \{x_1, \ldots, x_i\}$. Therefore

$$[G_{x_1,\ldots,x_i}: G_{x_1,\ldots,x_{i+1}}] = |X \setminus \{x_1, \ldots, x_i\}| = |X| - i.$$

The lemma then follows from the observation that

$$[G: G_A] = [G: G_{x_1}] \cdot [G_{x_i}: G_{x_1,x_2}] \ldots [G_{x_1,\ldots,x_{k-1}}: G_A].$$

For G k-fold transitive on X and A a k-element subset of X, let t_k denote the number of G_A-orbits of X. If G is $|X|$-fold transitive on X and $k > |X|$ we set $t_k = 0$.

THEOREM 5. *If G is k-fold transitive on X, then*

$$\chi_{k+1} = t_k - \sum_{i=1}^{k-1} S_1(k, i)\chi_{i+1}.$$

PROOF. From combinatorics and Theorem 2,

$$\sum_{|A|=k} \sum_{\sigma \in G_A} \chi(\sigma) = \binom{|X|}{k} \cdot t_k \cdot |G_A|.$$

This sum can also be obtained as

$$\sum_{\sigma \in G} \binom{\chi(\sigma)}{k} \chi(\sigma) = \frac{1}{k!} \sum_{i=1}^{k} S_1(k, i) \sum_{\sigma \in G} \chi^{i+1}(\sigma)$$

$$= \frac{|G|}{k!} \sum_{i=1}^{k} S_1(k, i)\chi_{i+1}.$$

If one equates these sums and uses Lemma 4 to simplify, the formula asserted in Theorem 5 is obtained.

THEOREM 6. *If G is k-fold transitive on X, then*

$$\chi_{k+1} = t_k + \sum{}' P_k(n_1, \ldots, n_j) \cdot t_{n_j}.$$

PROOF. By Theorem 2, $\chi_1 = t_0 = 1$. Assume then the assertion holds for $\chi_1, \ldots, \chi_k$. Then by Theorem 5,

$$\chi_{k+1} = t_k - \sum_{l=1}^{k-1} S_1(k, l)\left(t_l + \sum{}' P_l(n_1, \ldots, n_p)t_{n_p}\right)$$

$$= t_k - \sum_{l=1}^{k-1} S_1(k, l)t_l - \sum_{l=1}^{k-1} S_1(k, l) \sum{}' P_l(n_1, \ldots, n_p)t_{n_p}$$

$$= t_k + \sum{}' P_k(n_1, \ldots, n_j)t_{n_j}.$$

Therefore Theorem 6 follows by induction.

We now give formulas that express the χ_n in terms of Stirling numbers of the second kind.

THEOREM 7. *If G is $(k - 1)$-fold transitive on X, then*

$$\chi_k = \sum_{j=1}^{k} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} i!\, t_{k-1-i} S_2(k, j).$$

Moreover, if G is also k-fold transitive on X, then

$$\chi_k = \sum_{j=1}^{\min\{k,|X|\}} S_2(k,j).$$

PROOF. Define $\{a_n\}$ by $\chi_k = \sum_{n=0}^{k} a_n S_2(k, n)$. By Theorem 5,

$$\begin{aligned} t_n &= \sum_{i=1}^{n} S_1(n,i)\left(\sum_{j=1}^{i+1} a_j S_2(i+1,j)\right) \\ &= \sum_{j=1}^{n+1} a_j \sum_{i=1}^{n+1} S_1(n,i)(jS_2(i,j) + S_2(i,j-1)) \\ &= na_n + a_{n+1}. \end{aligned}$$

By Theorem 2, $a_1 = t_0 = 1$. Solving the difference equations $t_n = na_n + a_{n+1}$ yields

$$a_n = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} i!\, t_{n-1-i}.$$

This proves the first part of Theorem 7. If G is also k-fold transitive and $k < |X|$, then $t_i = i + 1$ for $0 \leqslant i \leqslant k$. If G is $|X|$-fold transitive, then $t_i = i + 1$ for $i < |X|$, $t_{|X|} = |X|$ and $t_i = 0$ for $i > |X|$. Thus

$$\chi_k = \sum_{i=0}^{k} a_i S_2(k,i) = \sum_{i=1}^{\min\{k,|X|\}} S_2(k,i).$$

A special case of interest is $G = \mathrm{Sym}(n)$ and $X = \{1, \ldots, n\}$. Here we choose to denote χ_k as $\chi(n, k)$. Thus

$$\chi(n,k) = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \chi^k(\sigma),$$

and Theorems 5, 6 and 7 become

$$(5') \qquad \sum_{i=1}^{k} S_1(k,i)\chi(n,i+1) = \begin{cases} k+1 & \text{if } k < n, \\ n & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

$$(6') \qquad \chi(n,k) = \begin{cases} k + \sum{}' P_{k-1}(n_1, \ldots, n_j)(n_j+1) & \text{if } k \leqslant n, \\ n + \sum{}' P_n(n_1, \ldots, n_j)(n_j+1) & \text{if } k = n+1, \\ \sum\limits_{n_j = n}{}'' P_{k-1}(n_1, \ldots, n_j)n + \sum\limits_{n_j < n}{}'' P_{k-1}(n_1, \ldots, n_j)(n_j+1) & \\ \qquad \text{if } k > n+1. \end{cases}$$

$$(7') \qquad \chi(n,k) = \sum_{i=1}^{\min\{n,k\}} S_2(k,i).$$

Note that by equating (6′) and (7′) we obtain an identity solely between Stirling numbers. We conclude by remarking that Otto G. Ruehr [10] has shown the sequence $\{\chi(n, n)\}$ satisfies

$$x \exp(e^x + x - 1) = \sum_{n=1}^{\infty} \frac{\chi(n, n)}{n!} x^n.$$

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