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A REDUCIBILITY CONDITION FOR RECURSIVENESS[1]

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ABSTRACT. A result due to Jockusch, equating recursiveness of a set to a reducibility condition on its jump, is sharpened.

Introduction. Unexplained notation is taken from Rogers [5]. In an unpublished proof in 1970, Carl Jockusch showed that if $A' \leqslant_{\mathrm{btt}} \varnothing'$ then A is recursive (the converse is immediate). A short proof of an indirect nature was later obtained by Gordon Phillips, a student of Jockusch. This paper gives a fairly direct proof of a more basic result from which that of Jockusch follows immediately.

We write $A \oplus B$ for the set $\{2x\colon x \in A\} \cup \{2x+1\colon x \in B\}$. Following Soare [6] set $H_A = \{e\colon W_e \cap A \neq \varnothing\}$. In the context of A co-r.e. Soare has called H_A the "weak jump" of A. For general A it seems appropriate to give this name to $H_A \oplus H_{\bar{A}}$ (if A is co-r.e. and nonempty then $H_A \equiv H_A \oplus H_{\bar{A}}$). The relationship of the weak jump and S-reducibility [2] is analogous to that of the jump and Turing reducibility. H_A has been studied by Hay [3], [4] and Soare [6], [7] and has been involved in a number of interesting relationships.

Let t be the tt-condition $\langle\langle x_1, \ldots, x_n\rangle, \alpha\rangle$ (see [5, p. 110]). We denote the associated set $\{x_1, \ldots, x_n\}$ by F_t. If $\alpha(0, \ldots, 0) = 0$ we say t is *zero-preserving*.

Results.

THEOREM 1. *If A is r.e. and $H_{\bar{A}} \leqslant_{\mathrm{btt}} \varnothing'$ then A is recursive.*

PROOF. Let n be the least integer such that $H_{\bar{A}} \leqslant_{\mathrm{btt}} \varnothing'$ with norm bounded by n. Let h be a recursive function such that $e \in H_{\bar{A}} \Leftrightarrow$ the tt-condition $h(e)$ is satisfied by $\varnothing'$, and each $h(e)$ has norm bounded by n. Assume A nonrecursive. Define

$$W_{f(e,x)} = \begin{cases} W_e & \text{if } x \in \varnothing', \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that if $x \in \varnothing'$ then $f(e,x) \in H_{\bar{A}} \Leftrightarrow e \in H_{\bar{A}}$. Define $W_{g(e,y)} = W_e \cup \{y\}$. If $y \in A$ then $g(e,y) \in H_{\bar{A}} \Leftrightarrow e \in H_{\bar{A}}$. Fix e. Set

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$$B_1 = \{f(e,x) \colon x \in \varnothing' \text{ and } F_{h(f(e,x))} \cap \varnothing' \neq \varnothing\},$$

$$B_2 = \{g(e,y) \colon y \in A \text{ and } F_{h(g(e,y))} \cap \varnothing' \neq \varnothing\}.$$

Note that B_1 and B_2 are r.e. Put $B = B_1 \cup B_2$. We will show that B is nonempty.

Observe that for any u, if $F_{h(u)} \cap \varnothing' = \varnothing$, then $u \in H_{\bar{A}} \Leftrightarrow h(u)$ is not zero-preserving. We distinguish two cases:

Case 1. $e \in H_{\bar{A}}$. Then $x \in \varnothing' \Leftrightarrow f(e,x) \in H_{\bar{A}}$. Thus if $F_{h(f(e,x))} \cap \varnothing' = \varnothing$ we have $x \in \bar{\varnothing}' \Leftrightarrow h(f(e,x))$ is zero-preserving.

Suppose $B_1 = \varnothing$. Then $F_{h(f(e,x))} \cap \varnothing' \neq \varnothing \Rightarrow x \in \bar{\varnothing}'$. Putting these together we get $x \in \bar{\varnothing}' \Leftrightarrow F_{h(f(e,x))} \cap \varnothing' \neq \varnothing$ or $h(f(e,x))$ is zero-preserving. This implies $\bar{\varnothing}'$ is r.e., a falsehood. Thus $B_1 \neq \varnothing$.

Case 2. $e \notin H_{\bar{A}}$. In this case $y \in A \Leftrightarrow g(e,y) \notin H_{\bar{A}}$. Consequently if $F_{h(g(e,y))} \cap \varnothing' = \varnothing$ we have $y \in \bar{A} \Leftrightarrow h(g(e,y))$ is not zero-preserving. Assuming $B_2 = \varnothing$ now gives $y \in \bar{A} \Leftrightarrow F_{h(g(e,y))} \cap \varnothing' \neq \varnothing$ or $h(g(e,y))$ is not zero-preserving. It follows that A is recursive, contrary to supposition. Here we conclude $B_2 \neq \varnothing$.

Now let z be the first element in an enumeration of B. Since $F_{h(z)} \cap \varnothing' \neq \varnothing$ we can form a tt-condition t with norm bounded by $n - 1$ such that t is satisfied by $\varnothing' \Leftrightarrow h(z)$ is satisfied by $\varnothing' \Leftrightarrow e \in H_{\bar{A}}$.

Redefining $h(e) = t$ we see that $H_{\bar{A}} \leqslant_{\text{btt}} \varnothing'$ with norm bounded by $n - 1$, contradicting the minimality of n. We conclude that A is recursive. Q.E.D.

Note that the above proof does not supply a decision procedure for A. Theorem 1 confirms a conjecture of Hay [3].

THEOREM 2. *If $H_A \oplus H_{\bar{A}} \leqslant_{\text{btt}} \varnothing'$, then A is recursive.*

PROOF. If $H_A \oplus H_{\bar{A}} \leqslant_{\text{btt}} \varnothing'$ then $A \leqslant_{\text{btt}} \varnothing'$.

By [5, Theorem 14-IX] A is a Boolean combination of r.e. sets. It follows from Ershov [1] that there are r.e. sets $R_1, \ldots, R_n$ such that $R_1 \subseteq \cdots \subseteq R_n$ and

$$A = \begin{cases} \bigcup_{i=1}^{n/2} (R_{2i} - R_{2i-1}) & \text{if } n \text{ is even,} \\ R_1 \cup \bigcup_{i=1}^{(n-1)/2} (R_{2i+1} - R_{2i}) & \text{if } n \text{ is odd.} \end{cases}$$

We will prove the theorem by induction on n. For $n = 1$ the result follows from Theorem 1. Suppose $n > 1$. Let f enumerate R_n. Let $B = f^{-1}(\bar{A})$. Then $B \leqslant_m \bar{A}$. Hence $H_B \leqslant_1 H_{\bar{A}}$ and $H_{\bar{B}} \leqslant_1 H_A$ and so $H_B \oplus H_{\bar{B}} \leqslant_{\text{btt}} \varnothing'$.

Let $S_i = f^{-1}(R_i)$, $1 \leqslant i \leqslant n - 1$. Then

$$B = \begin{cases} \bigcup_{i=1}^{(n-1)/2} (S_{2i} - S_{2i-1}) & \text{if } n - 1 \text{ is even,} \\ S_1 \cup \bigcup_{i=1}^{(n-2)/2} (S_{2i+1} - S_{2i}) & \text{if } n - 1 \text{ is odd.} \end{cases}$$

The inductive hypothesis now yields B is recursive. It follows that A is r.e. By Theorem 1, A is recursive. Q.E.D.

COROLLARY (JOCKUSCH). *If* $A' \leqslant_{\text{btt}} \varnothing'$ *then* A *is recursive.*

PROOF. Clearly $H_A \leqslant_1 A'$ and $H_{\bar{A}} \leqslant_1 A'$. The result follows.

Closing remarks. In view of Theorem 2, it might be supposed that $H_A \leqslant_{\text{btt}} \varnothing' \Leftrightarrow A$ r.e. This is false, as is demonstrated by an elaborate construction in [3].

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