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PIECEWISE LINEAR FUNCTIONS WITH ALMOST ALL POINTS EVENTUALLY PERIODIC

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ABSTRACT. Let f: $[0, 1] \to [0, 1]$ be continuous, and let f^p denote the pth iterate of f. Li and Yorke [2] proved that if there is a point $x \in [0, 1]$ such that $f^3(x) = x$ but $f(x) \neq x$, then f is chaotic in the sense that f has periodic points of arbitrarily large period, and uncountably many points which are not even asymptotically periodic. But this chaos can be measure theoretically trivial. For each $p \geqslant 3$ we construct a continuous, piecewise linear function f: $[0, 1] \to [0, 1]$ such that f is chaotic, but almost every point of $[0, 1]$ has eventual period p. The condition "eventual period p" cannot be replaced by "period p". We prove that if $f^p(x) = x$ for almost all $x \in [0, 1]$, then $f^2(x) = x$ for all $x \in [0, 1]$. Moreover, we describe a normal form for all such "square roots of the identity."

Let f: $[0, 1] \to [0, 1]$ be continuous. The iterates of f are defined as follows: $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for $n = 1, 2, 3, \ldots$. The point $x \in [0, 1]$ is periodic under f with period p if $f^p(x) = x$ but $f^k(x) \neq x$ for $k = 1, 2, \ldots, p - 1$. If $f^n(x)$ has period p for some n, then x is eventually periodic under f with period p. Li and Yorke [2] have recently obtained the remarkable result that if f: $[0, 1] \to [0, 1]$ has a point of period three, then f is "chaotic" in the sense that, first, there are points $x \in [0, 1]$ of arbitrarily large period (in fact, of all periods), and, second, there is an uncountable set $S \subset [0, 1]$ such that no point of S is even asymptotically periodic (that is, if $y \in S$ and if $x \in [0, 1]$ is periodic, then $\limsup |f^n(y) - f^n(x)| > 0$), and such that, if y_1, y_2 are any two points of S, then $\liminf |f^n(y_1) - f^n(y_2)| = 0$ and $\limsup |f^n(y_1) - f^n(y_2)| > 0$. More generally, Li and Yorke proved that if there is a point $x \in [0, 1]$ such that either $f^3(x) \leqslant x < f(x) < f^2(x)$ or $f^3(x) \geqslant x > f(x) > f^2(x)$, then f is chaotic. By a combinatorial argument, Nathanson [7] extended this result to show that if f has a point of period five or seven, then f is chaotic. Ulam, May, Oster, and others [1], [3]–[6], [8] have studied in detail the iterations of nonlinear functions f and the dependence of the trajectories $x, f(x), f^2(x), f^3(x), \ldots$ on the initial value x.

The object of this note is to show that, from the point of view of Lebesgue measure, the results on chaos can be misleading. For every $p \geqslant 3$ we shall construct a continuous, piecewise linear function f: $[0, 1] \to [0, 1]$ such that almost every $x \in [0, 1]$ has eventual period p. Moreover, f will be chaotic. This

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result is best possible in the sense that the word "eventual" cannot be removed from the statement of the theorem. For if almost every point $x \in [0,1]$ has period p under f, then the continuity of f implies that $f^p(x) = x$ for all x, and so $f^p =$ identity. However, we shall prove that if $f^p =$ identity, then $f^2 =$ identity. Moreover, we shall describe a normal form for all such square roots of the identity.

THEOREM 1. *Let* $p \geqslant 3$ *and let* $\delta \in (0, 2^{-p})$. *Define* f: $[0,1] \to [0,1]$ *in the following way*:

$$f(x) = \begin{cases} x + 1/p, & 0 \leqslant x \leqslant (p-1)/p, \\ 1 - (1-\delta)\delta^{-1}(x - (p-1)/p), & (p-1)/p < x < (p-1)/p + \delta, \\ x - (p-1)/p, & (p-1)/p + \delta \leqslant x \leqslant 1. \end{cases}$$

Then f is continuous, piecewise linear, chaotic, and almost every point $x \in [0,1]$ *has eventual period p under f.*

PROOF. Clearly, f is continuous and piecewise linear. By the theorem of Li and Yorke, f is also chaotic, since

$$f^3\left(\frac{p-2}{p}\right) = \frac{1}{p} \leqslant \frac{p-2}{p} < f\left(\frac{p-2}{p}\right) = \frac{p-1}{p} < f^2\left(\frac{p-2}{p}\right) = 1.$$

Let $C = \bigcup_{i=1}^{p}[(i-1)/p + \delta, i/p]$. For $i = 1, 2, \ldots, p-1$, the function f maps the interval $[(i-1)/p + \delta, i/p]$ linearly onto $[i/p + \delta, (i+1)/p]$ by the rule $f(x) = x + 1/p$. Also, f maps the interval $[(p-1)/p + \delta, 1]$ linearly onto the interval $[\delta, 1/p]$ by the rule $f(x) = x - (p-1)/p$. Thus, each point $x \in C$ has period p, and $f(C) = C$. Let $x \in [0,1]$. If $f^m(x) \in C$ for some m, then $f^n(x) \in C$ for all $n \geqslant m$, and x has eventual period p. We define

$$C^* = \{x \in [0,1] | f^m(x) \in C \text{ for some } m\},$$

$$U^* = \{x \in [0,1] | f^n(x) \notin C \text{ for } n = 0, 1, 2, 3, \ldots\}.$$

The sets C^* and U^* partition $[0,1]$. Every point of C^* has eventual period p. If $x \in [0,1]$ does not have eventual period p, then $u \in U^*$. Clearly, $U^* \subset \bigcup_{i=0}^{p-1}(i/p, i/p + \delta) \cup \{0\}$. Let $\mu(X)$ denote the Lebesgue measure of X. We shall prove that $\mu(C^*) = 1$, or, equivalently, that $\mu(U^*) = 0$.

We begin by studying the open interval $U_0 = ((p-1)/p, (p-1)/p + \delta)$. Let $U_n = \{x \in U_0 | f^n(x) \notin C\}$. Clearly, $U_0 \supset U_1 \supset U_2 \supset U_3 \supset \cdots$.

Let $\lambda = \delta/(1-\delta)$. We shall prove, by induction on n, that each U_n is a union of disjoint open intervals whose lengths are of the form $\delta\lambda^k$ for $k = 0, 1, 2, \ldots, n$, and that f^n maps each of these intervals linearly onto one of the $p-1$ intervals $(i/p, i/p + \delta)$ for $i = 1, 2, \ldots, p-1$. Moreover, if $A_{k,n}^{(i)}$ denotes the number of open intervals of length $\delta\lambda^i$ of U_n which f^n maps onto $(i/p, i/p + \delta)$, then the integers $A_{k,n}^{(i)}$ can be computed by the following rules:

$$A_{0,0}^{(i)} = \begin{cases} 1 & \text{if } i = p-1, \\ 0 & \text{if } i = 1, 2, \ldots, p-2; \end{cases}$$

$$A_{k,n}^{(i)} = \sum_{j=1}^{i} A_{k-1,n-j}^{(p-1)} \quad \text{for } n = 1, 2, \ldots,$$

where $A_{k,n}^{(i)} = 0$ if $k < 0$ or $n < 0$.

These statements are obviously true for $n = 0$. Let us assume that they hold for some $n - 1 \geqslant 0$. We want to describe the structure of U_n. Since $U_n \subset U_{n-1}$, it is enough to understand how f^n acts on the intervals that make up U_{n-1}. Let I be an open interval of U_{n-1} of length $\delta\lambda^k$, where $k \leqslant n-1$. If f^{n-1} maps I linearly onto $(i/p, i/p + \delta)$ for some $i = 1, 2, \ldots, p-2$, then f^n maps I linearly onto $((i+1)/p, (i+1)/p + \delta)$. If f^{n-1} maps I linearly onto $((p-1)/p, (p-1)/p + \delta)$, then f^n maps I linearly onto $(\delta, 1)$. Since the length of I is $\delta\lambda^k$, it follows that the slope of f^n on I has absolute value $(1-\delta)/\delta\lambda^k = 1/\lambda^{k+1}$. Moreover, for each $i = 1, 2, \ldots, p-1$ there is exactly one open interval of length $\delta\lambda^{k+1}$ of I which f^n maps linearly onto $(i/p, i/p + \delta)$. The function f^n sends the complement of these $p-1$ intervals into C. It follows that $A_{k,n}^{(1)} = A_{k-1,n-1}^{(p-1)}$ and $A_{k,n}^{(i)} = A_{k,n-1}^{(i-1)} + A_{k-1,n-1}^{(p-1)}$ for $i = 2, 3, \ldots, p-1$. These relations imply that $A_{k,n}^{(i)} = \sum_{j=1}^{i} A_{k-1,n-j}^{(p-1)}$. This completes the induction.

Now we can compute the measure of the sets U_n. It follows from the definition of the numbers $A_{k,n}^{(i)}$ that

$$\mu(U_n) = \sum_{i=1}^{p-1} \sum_{k=0}^{n} A_{k,n}^{(i)} \delta\lambda^k = \delta \sum_{i=1}^{p-1} P_n^{(i)}(\lambda),$$

where $P_n^{(i)}(x)$ is the polynomial defined by

$$P_n^{(i)}(x) = \sum_{k=0}^{n} A_{k,n}^{(i)} x^k.$$

The recurrence relations for the coefficients $A_{k,n}^{(i)}$ imply that

$$P_0^{(i)}(x) = \begin{cases} 1 & \text{if } i = p-1, \\ 0 & \text{if } i = 1, 2, \ldots, p-2; \end{cases}$$

$$P_n^{(i)}(x) = x \sum_{j=1}^{i} P_{n-j}^{(p-1)}(x) \quad \text{for } n = 1, 2, \ldots,$$

where $P_n^{(i)}(x) = 0$ for $n < 0$.

Clearly, the degree of $P_n^{(i)}(x)$ is n. Write n in the form $n = q(p-1) - r$, where $r = 0, 1, 2, \ldots, p-2$. I claim that $P_n^{(p-1)}(x)$ is divisible by x^q. This is certainly true for $q = 0$ and $q = 1$, since $P_n^{(p-1)}(x)$ is divisible by x for $n = 1, 2, \ldots, p-1$. Moreover, the recurrence relation implies that if x^k divides $P_m^{(p-1)}(x)$, then x^k divides $P_n^{(p-1)}(x)$ for all $n \geqslant m$. Let us assume the claim is true for some $q - 1 \geqslant 1$ and $r = 0, 1, \ldots, p-2$. If $n = q(p-1) - r$, then $n - (p-1) = (q-1)(p-1) - r$, and so $P_{n-(p-1)}^{(p-1)}(x)$ is divisible by

x^{q-1}. Consequently, $P_{n-j}^{(p-1)}(x)$ is divisible by x^{q-1} for $j = 1, 2, \ldots, p-1$. Since $P_n^{(p-1)}(x) = x\sum_{j=1}^{p-1} P_{n-j}^{(p-1)}(x)$, it follows that $P_n^{(p-1)}(x)$ is divisible by $x \cdot x^{q-1} = x^q$. The claim follows by induction on q.

Since the coefficients of $P_n^{(p-1)}(x)$ are nonnegative, it follows that for $0 < \lambda < 1$ we have

$$P_n^{(i)}(\lambda) \leqslant P_n^{(p-1)}(\lambda) \leqslant \lambda^{(n+r)/(p-1)} P_n^{(p-1)}(1) \leqslant \lambda^{n/(p-1)} P_n^{(p-1)}(1)$$

where $i = 1, 2, 3, \ldots, p-1$ and $n = q(p-1) - r$. But the integers $P_n^{(p-1)}(1)$ satisfy the recurrence relations

$$P_0^{(p-1)}(1) = 1, \qquad P_n^{(p-1)}(1) = \sum_{j=1}^{p-1} P_{n-j}^{(p-1)}(1) \quad \text{for } n = 1, 2, \ldots,$$

where $P_n^{(p-1)}(1) = 0$ for $n < 0$. An easy induction shows that $P_n^{(p-1)}(1) \leqslant 2^n$ for $n = 0, 1, 2, \ldots$. Therefore,

$$P_n^{(i)}(\lambda) \leqslant (2\lambda^{1/(p-1)})^n.$$

But for $0 < \delta < 2^{-p}$ we have

$$0 < \lambda^{1/(p-1)} = \left(\frac{\delta}{1-\delta}\right)^{1/(p-1)} < (2\delta)^{1/(p-1)} < \frac{1}{2}$$

and so $0 < 2\lambda^{1/(p-1)} < 1$. Therefore,

$$\mu(U_n) = \delta \sum_{i=1}^{p-1} P_n^{(i)}(\lambda) \leqslant \delta(p-1)(2\lambda^{1/(p-1)})^n$$

Consequently,

$$\lim_{n\to\infty} \mu(U_n) = 0.$$

Let us return to the set $U^* = \{x \in [0,1] | f^n(x) \notin C \text{ for } n = 0, 1, 2, \ldots\}$. Let $U_n^* = \{x \in [0,1] | f^n(x) \notin C\}$. Then

$$U_0^* = \bigcup_{i=0}^{p-1}\left(\frac{i}{p}, \frac{i}{p} + \delta\right) \cup \{0\} \supset U_1^* \supset U_2^* \supset \cdots$$

and $U^* = \bigcap_{n=0}^{\infty} U_n^*$. Therefore, $\mu(U^*) = \lim_{n\to\infty} \mu(U_n^*)$, Since $f(0) = 1/p \in C$, we have $0 \notin U_n^*$ for $n \geqslant 1$. For $i = 0, 1, \ldots, p-1$, let $U_n^{(i)} = \{x \in (i/p, i/p + \delta) | f^n(x) \notin C\}$. Then $U_n^{(p-1)} = U_n$ and $U_n^* = \bigcup_{i=0}^{p-1} U_n^{(i)}$.

For $i = 0, 1, 2, \ldots, p-2$, the map f^{p-1-i} sends $U_0^{(i)} = (i/p, i/p + \delta)$ linearly onto $((p-1)/p, (p-1)/p + \delta) = U_0^{(p-1)} = U_0$ according to the rule $f^{p-1-i}(x) = x + (p-1-i)/p$. Therefore,

$$U_n^{(i)} = \{x - (p-1-i)/p | x \in U_{n-p+1+i}\}$$

for $n \geqslant p-1-i$, and so $\mu(U_n^{(i)}) = \mu(U_{n-p+1+i})$. Since $U_n^* = \bigcup_{i=0}^{p-1} U_n^{(i)}$, we have

$$\mu(U_n^*) = \sum_{i=0}^{p-1} \mu(U_n^{(i)}) = \sum_{i=0}^{p-1} \mu(U_{n-p+1+i})$$

and so

$$\mu(U^*) = \lim_{n\to\infty} \mu(U_n^*) = 0.$$

This completes the proof of the theorem.

THEOREM 2. *If f: $[0,1] \to [0,1]$ is a continuous function such that $f^p(x) = x$ for all x, then $f^2(x) = x$ for all x. In particular, if p is odd, then $f(x) = x$ for all x.*

PROOF. If $f^p(x) = x$ for all x, then f is a continuous bijection of $[0,1]$, and so f is monotone and either $f(0) = 0, f(1) = 1$ or $f(0) = 1, f(1) = 0$.

Let f be a monotone function such that $f(0) = 0, f(1) = 1$. If $f(x) \neq x$ for some $x \in (0,1)$, say, $f(x) > x$, then there is an interval $[a,b]$ with $0 \leqslant a < x < b \leqslant 1$ such that $a < x < f(x) < b$ for all $x \in (a,b)$. Then

$$0 \leqslant a < x < f(x) < f^2(x) < \cdots < f^{p-1}(x) < f^p(x) < \cdots < b \leqslant 1$$

and so $f^p(x) \neq x$. Therefore, if $f^p(x) = x$ for all x, and $f(0) = 0, f(1) = 1$, then $f(x) = x$ for all x.

Let f be a monotone function such that $f(0) = 1, f(1) = 0$. If p is odd, then $f^p(0) = 1$. Therefore, if $f^p(x) = x$ for all x, then $p = 2q$ is even. Let $g(x) = f^2(x)$. Then g is a monotone function such that $g(0) = 0, g(1) = 1$, and $g^q(x) = f^p(x) = x$. Therefore, $g(x) = f^2(x) = x$ for all $x \in [0,1]$. This proves the theorem.

The next result shows that all square roots of the identity are obtained by conjugating the function $h(x) = 1 - x$ by an increasing, "half-linear" function γ. This observation is due to David Kazhdan.

THEOREM 3. *Let f: $[0,1] \to [0,1]$ be a continuous function such that $f(0) = 1$, $f(1) = 0$, and $f^2(x) = x$ for all $x \in [0,1]$. Then there is a unique increasing function γ: $[0,1] \to [0,1]$ with γ linear on $[0,\frac{1}{2}]$ such that*

$$f(x) = \gamma(1 - \gamma^{-1}(x)) \tag{1}$$

for all $x \in [0,1]$.

PROOF. Clearly, f is a monotone decreasing function on $[0,1]$. Let $a \in (0,1)$ be a fixed point of f. If $x < a$, then $f(x) > f(a) = a > x$. If $x > a$, then $f(x) < f(a) = a < x$. Therefore, a is the unique fixed point of f.

We define the function γ on $[0,1]$ in the following way:

$$\gamma(x) = \begin{cases} 2ax, & 0 \leqslant x \leqslant \frac{1}{2}, \\ f(2a(1-x)), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Observe that $f(2a(1-x))$ increases monotonically from a to 1 as x increases from $\frac{1}{2}$ to 1. Therefore, γ has a continuous inverse on $[0,1]$, and $\gamma^{-1}(x) = x/2a$ for $x \in [0,a]$.

Suppose $0 \leqslant x \leqslant a$. Then $1 - \gamma^{-1}(x) = 1 - x/2a \in [\frac{1}{2}, 1]$, and so

$$\gamma(1-\gamma^{-1}(x)) = \gamma(1 - x/2a) = f(2a(1-(1-x/2a))) = f(x).$$

Suppose $a \leqslant x \leqslant 1$. Then

$$\gamma^{-1}(x) = y \in \left[\frac{1}{2}, 1\right] \quad \text{and} \quad x = \gamma(y) = f(2a(1-y)).$$

Therefore, $f(x) = f^2(2a(1-y)) = 2a(1-y)$. On the other hand, $1 - y \in [0, \frac{1}{2}]$ and so

$$\gamma(1-\gamma^{-1}(x)) = \gamma(1-y) = 2a(1-y) = f(x).$$

This proves that $f(x) = \gamma(1-\gamma^{-1}(x))$ for all $x \in [0,1]$.

Let δ: $[0,1] \to [0,1]$ be linear on $[0,\frac{1}{2}]$ and satisfy $f(0) = 0$ and

(2) $$f(x) = \delta(1-\delta^{-1}(x))$$

for all $x \in [0,1]$. Let $\delta(\frac{1}{2}) = b$. Then

$$f(b) = \delta(1-\delta^{-1}(b)) = \delta(\tfrac{1}{2}) = b$$

and so b is a fixed point of f. But f has the unique fixed point a. Therefore, $a = b$ and $\delta(x) = 2ax = \gamma(x)$ for $x \in [0,\frac{1}{2}]$.

If we replace x by $\gamma(x)$ in (1) and (2), we obtain

$$\gamma(1-x) = \delta(1-\delta^{-1}\gamma(x))$$

for all $x \in [0,1]$. Suppose $\frac{1}{2} \leqslant x \leqslant 1$. Then $\gamma(x) \in [a,1]$ and $\delta^{-1}\gamma(x) \in [\frac{1}{2}, 1]$. Therefore,

$$2a(1-x) = \gamma(1-x) = \delta(1-\delta^{-1}\gamma(x)) = 2a(1-\delta^{-1}\gamma(x))$$

and so $x = \delta^{-1}\gamma(x)$ and $\gamma(x) = \delta(x)$ for $x \in [\frac{1}{2}, 1]$. This proves the theorem.

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