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BAIRE* 1, DARBOUX FUNCTIONS

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ABSTRACT. It is well known that a function f: $[0, 1] \to R$ is Baire 1 if and only if in any closed set C there is a point x_0 at which the restricted function $f|C$ is continuous. Functions will be called Baire* 1 if they satisfy the following stronger property: For every closed set C there is an open interval (a, b) with $(a, b) \cap C \neq \varnothing$ such that $f|C$ is continuous on (a, b). Functions which are both Baire* 1 and Darboux are examined. It is known that approximately derivable functions are Baire* 1. Among other things it is shown here that L_p-smooth functions are Baire* 1. A new result about the L_p-differentiability of L_p-smooth, Darboux functions is shown to follow immediately from the main properties of Baire* 1, Darboux functions.

It is well known that a function f: $[0, 1] \to R$ is Baire 1 if and only if in any closed set C there is a point x_0 at which the restricted function $f|C$ is continuous. In this paper functions will be called Baire* 1 if they satisfy the following stronger property: For every closed set C there is an open interval (a, b) with $(a, b) \cap C \neq \varnothing$ such that $f|C$ is continuous on (a, b). Functions which are both Baire* 1 and Darboux are the main topic of this paper. It is known [6] that approximately derivable functions are Baire* 1. Among other things it is shown here that L_p-smooth functions are Baire* 1. A new result about the L_p-differentiability of L_p-smooth, Darboux functions is shown to follow immediately from the main properties of Baire* 1, Darboux functions.

The following conventions and notations will be used. All functions will be real valued and defined on $[0, 1]$. A component of a set V is a maximal subinterval of V; it will be denoted as (a, b) even if a or b belongs to V.

If f is a Baire* 1, Darboux function and C is the set of points at which f is continuous, then the interior of C will be dense. This dense set will be denoted as $U(f)$ or simply U and its complement as $P(f)$ or simply P. Finally, the function f restricted to a set Q will be denoted as $f|Q$.

The first theorem is rather simple but plays an essential role in the later theorems.

THEOREM 1. *Let f be Baire* 1, Darboux. If $U \neq [0, 1]$ there is a component of U on which f is not monotone.*

PROOF. Assume instead that f is monotone on each component (a, b) of U. Then, in addition:

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(1) $\lim_{h\to 0^+} f(a+h) = f(a)$ and $\lim_{h\to 0^+} f(b-h) = f(b)$, because of the Darbouxness of f.

(2) Further, $\min(f(a), f(b)) \leqslant f(x) \leqslant \max(f(a), f(b))$, for all x in (a, b).

Since $P \neq \varnothing$ it is possible to select an open interval (c, d) with $(c, d) \cap P \neq \varnothing$ and $f|P$ continuous on (c, d). The continuity of $f|P$ on (c, d) combined with (1) and (2) gives that f is actually continuous on (c, d), contradicting $(c, d) \cap P \neq \varnothing$.

Theorem 1 can be thought of as providing a test whereby Baire 1, Darboux functions can be examined for monotonicity. More precisely, Theorem 1 leads immediately to the following result similar to that of Bruckner [2].

COROLLARY 1. *Let T be any function-theoretical property sufficiently strong that*:

(1) *Any Baire* 1, *Darboux function having T is Baire** 1.

(2) *Any continuous function having T is monotone.*

Then any Baire 1, *Darboux function having T is monotone.*

In [3] Croft constructed an example of a Baire 1, Darboux function which is zero almost everywhere but is not identically zero. Theorem 1, however, shows that if a Baire* 1 function is zero on a dense set then it is identically zero. Moreover, for Croft's function f let C be the points at which f is continuous. It is clear that $f(C) = \{0\}$ while $f([0, 1]) = J$, a nondegenerate interval. Therefore, $f(C)$ is not dense in J. Theorems 2 and 4 below show that the situation is much better for Baire* 1, Darboux functions.

THEOREM 2. *Let f be Baire** 1, *Darboux. Then $f([0, 1]) \setminus f(U)$ is nowhere dense.*

PROOF. If f is identically a constant there is nothing to prove. It will be assumed therefore that $f([0, 1]) = I$ is a nondegenerate interval. Let (c, d) be a subinterval of I. It will be shown that there is an open interval (a, b) in U whose image J under f is a nondegenerate interval intersecting (c, d).

Consider the function $g(x) = \min[d, \max(f(x), c)]$. This new function is Baire* 1, Darboux. Moreover, $f(x) = g(x)$ if $c < f(x) < d$. Therefore, $g(x)$ is not a constant function. Consider $U(g)$. Theorem 1 guarantees that there is some component (r, s) of $U(g)$ on which g is not constant. Since $g(x)$ is not constant on (r, s) and $c \leqslant g(x) \leqslant d$, there is some point x_0 with (i) $r < x_0 < s$, (ii) $c < g(x_0) < d$, and (iii) g nonconstant in any neighborhood of x_0. Now g is continuous on (r, s), and hence (r, s) has a subinterval (r_1, s_1) containing x_0 with $c < g(x_0) < d$ on (r_1, s_1). On this subinterval $f(x) = g(x)$. Therefore, $(r_1, s_1) \subset U(f)$, $J = f((r_1, s_1)) \subset (c, d)$, and f is not constant on (r_1, s_1). This completes the proof.

Theorem 2 implies the following:

COROLLARY 2. *Let f be Baire** 1, *Darboux. Let I be any interval. If $\{x\colon a < f(x) < b\} \cap I \neq \varnothing$, then $\{x\colon a < f(x) < b\} \cap I \cap U \neq \varnothing$. Thus Baire** 1, *Darboux functions have the Denjoy-Clarkson property.*

It is clear from Theorem 1 that any Baire* 1, Darboux function f must attain a local extremum at some point of U. However, it is easy to construct examples of Baire* 1, Darboux functions which have no absolute extrema and attain on U only local minima. The following theorem shows that local maxima must also occur somewhere in $[0, 1]$.

THEOREM 3. *Let f be Baire* 1, Darboux. If $E = \{x: f(x) > 0\}$ is not empty then there is a point x_0 in E at which f has a local maximum.*

PROOF. It is clear that only the case where $U \neq [0, 1]$ need be proven. Also, it is sufficient to prove the theorem for nonnegative functions. (If f is not nonnegative it could be replaced by $g(x) = \max(f(x), 0)$.) Corollary 2 insures that every open interval I with $I \cap E \neq \varnothing$ has $I \cap E \cap U \neq \varnothing$. Let (a, b) be any component of U with $V = (a, b) \cap E \neq \varnothing$. If it is assumed that f does not have a local maximum greater than zero inside (a, b) then V cannot contain a component of the form (c, d) with $a < c < d < b$. Thus the behavior of f on $[a, b]$ can be described in one of the following ways:

(1) $V = (a, c), c < b$. Then $f(a) > 0, a \neq b, f \equiv 0$ on $[c, b]$, and f is strictly decreasing on $[a, c]$.

(2) $V = (c, b), a < c$. Then $f(b) > 0, b \neq 1, f \equiv 0$ on $[a, c]$, and f is strictly increasing on $[c, b]$.

(3) $V = (a, b), f(a) > 0, f(b) > 0$, and f is strictly monotone on $[a, b]$.

(4) $V = (a, b), f(a) > 0, f(b) > 0$, and there is $a < c < b$ with f strictly decreasing on $[a, c]$ and strictly increasing on $[c, b]$.

(5) $V = (a, c) \cup (d, b), a < c \leqslant d < b, f(a) > 0, f(b) > 0$, and f strictly decreasing on $[a, c], f \equiv 0$ on $[c, d], f$ strictly increasing on $[d, b]$.

It should be noted that Theorem 1 implies that every open interval I with $I \cap P \neq \varnothing$ has nonempty intersection with components of U satisfying (4) or (5). Further, all five cases imply $E \cap P \neq \varnothing$. Let Q be the closure of $E \cap P$. Let (r, s) be an open interval with $(r, s) \cap Q \neq \varnothing$ and $f|Q$ continuous on (r, s). It can be assumed that r and s are elements of components of U satisfying (4) or (5). Let (a_1, b_1) be the component of U containing r and (a_2, b_2) that containing s. It may be assumed that f is strictly increasing on $[r, b_1]$ and strictly decreasing on $[a_2, s]$. Further, $f(a_2) > 0$ and $f(b_1) > 0$.

Since the set $Q \cap (r, s)$ is compact and $f|Q$ is continuous on (r, s), there is an x_0 in Q with $f(x_0) \geqslant f(x)$ for all x in $Q \cap (r, s)$. Since $E \cap P$ is dense in $Q, f(x_0) > 0$. It is further claimed that $f(x_0) \geqslant f(x)$ for all x in (r, s). To see this, let x_1 belong to $(r, s) \setminus Q$. Then either x_1 belongs to U or x_1 belongs to $P \setminus E$. If x_1 belongs to $U \cap (r, s)$ let I be the component of U containing x_1. One of the five cases above describes the behavior of f on I. In any of these five cases, it is not hard to see that at least one endpoint e of I belongs to $Q \cap (r, s)$ and $f(e) \geqslant f(x_1)$. If x_1 belongs to $P \setminus E$ then $f(x_1) \leqslant 0 < f(x_0)$. Thus f has a local maximum at x_0. This completes the proof.

COROLLARY 3. *Let f be Baire* 1, Darboux on an open interval containing* $[0, 1]$. *If*

$$\limsup_{h\to 0^+} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \geqslant 0$$

for all x in [0, 1], *then f is convex.*

PROOF. Same as [4].

THEOREM 4. *Let f: $[0, 1] \to R$ be Baire* 1, Darboux. Let C be the set of points at which f is continuous. Then $f([0, 1]) \setminus f(C)$ is at most countable.*

PROOF. Let E be the union of those open intervals I with $f(I) \setminus f(C)$ at most countable. Then the set E is open. Moreover, $U(f)$ is a subset of E. Thus E is nonempty, and if J is a component of E, then $f(J) \setminus f(C)$ is at most countable. It is necessary to show that $E = [0, 1]$. Assume instead that $H = [0, 1] \setminus E$ is nonempty. Then H is a closed nowhere dense set. Select an open interval (c, d), having endpoints in E, with $(c, d) \cap H \neq \varnothing$ and $f|H$ continuous on (c, d). The interval (c, d) is the disjoint union of the sets $(c, d) \cap H$ and $(c, d) \cap E$. Consider $V = (c, d) \cap E$. This is an open set. Then let $V = \bigcup_{1 \leqslant n < \infty}(a_n, b_n)$ and $f(V) = \bigcup_{1 \leqslant n < \infty} f((a_n, b_n))$. For each n, $f((a_n, b_n)) \setminus f(C)$ is at most countable. Therefore, $f(V) \setminus f(C)$ is at most countable. Let D be the set of points of discontinuity of f, and consider $(c, d) \cap H$. This is the disjoint union of $A_1 = (c, d) \cap H \cap C$ and $A_2 = (c, d) \cap H \cap D$. Obviously, $f(A_1) \setminus f(C) = \varnothing$. Finally, let x_0 belong to A_2. Since $f|H$ is continuous at x_0 and f is discontinuous at x_0, there is a sequence x_m from E converging to x_0, and an $\varepsilon > 0$ such that $|f(x_m) - f(x_0)| > \varepsilon$. It may be assumed that $f(x_m) > f(x_0) + \varepsilon$ and $x_m > x_0$ for all m. Let a_m be the left endpoint of the component of E containing x_m. (It may be that $a_m = x_0$.) Then $f(a_m)$ converges to $f(x_0)$. Let $\delta > 0$ be given. There is an M such that $m > M$ implies that $f(a_m) < f(x_0) + \delta$. Since $f(x_m) > f(x_0) + \varepsilon$ and f has the Darboux property on (a_m, x_m), the interval $(f(x_0) + \delta, f(x_0) + \varepsilon)$ is a subset of $f(V)$ for each $\delta > 0$. Therefore, $(f(x_0), f(x_0) + \varepsilon) \subset f(V)$. This means that either $f(x_0)$ is contained in $f(V)$ or is an endpoint of the interior of $f(V)$. There are only countably many such endpoints. Hence $f(A_2) \setminus f(V)$ is at most countable. Since $f(c, d) = f(V) \cup f(A_1) \cup f(A_2)$, the above facts show that $f(c, d) \setminus f(C)$ is at most countable. Thus $(c, d) \subset E$, contradicting $(c, d) \cap H \neq \varnothing$.

In the next section of the paper L_p-smooth functions are considered. It will be pointed out that such functions are Baire* 1. The proof of this fact is a matter of reinterpretation and rearrangement of results in [5]. Finally, it is shown that Theorem 3 of this paper can be used to obtain a new result about the L_p-differentiability properties of L_p-smooth, Darboux functions.

DEFINITION. A measurable function f: $[0, 1] \to R$ is L_p-smooth, $p \geqslant 1$, if for each x in $(0, 1)$

$$\left\{ \frac{1}{h} \int_0^h |f(x+t) + f(x-t) - 2f(x)|^p \, dt \right\}^{1/p} = o(h) \quad \text{as } h \to 0.$$

DEFINITION. A measurable function has at x_0 a first L_p-derivative, $L_p f'(x_0)$, provided there is an a_0 such that

$$\left\{ \frac{1}{2h} \int_{-h}^{h} |f(x_0 + t) - a_0 - L_p f'(x_0) t|^p \, dt \right\}^{1/p} = o(h) \quad \text{as } h \to 0.$$

The following lemma, in slightly different form, can be found in [1, p. 53]. It will be needed in connection with Theorem 5. However, it is of independent interest.

LEMMA 1. *Let* $\Sigma f_n(x) = f(x)$ *for all* x *in* $[0, 1]$ *with* $f_n(x)$ *continuous for* $n = 1, 2, \ldots$. *Let* $\Sigma a_n < +\infty$, *and* $a_n \geqslant 0$ *for* $n = 1, 2, \ldots$. *Suppose that for each* x *in* $[0, 1]$ *there is an* $N(x)$ *such that* $n > N(x)$ *implies that* $|f_n(x)| \leqslant a_n$. *Then for each closed set* C *there is an interval* (a, b), *with* $(a, b) \cap C \neq \varnothing$, *such that the series of functions converges uniformly on* $(a, b) \cap C$. *Hence* f *is Baire** 1.

PROOF. See Auerbach [1, p. 53]. In [1] the closed set C is assumed to be an interval. However, the proof given applies equally well to any closed set.

THEOREM 5. *Let* f *be* L_p*-smooth. Then* f *is Baire** 1.

PROOF. Suppose first that f is integrable. Then a perusal of the proof of Lemma 6 [5] and the lemma given above shows that f is Baire* 1.

In the general case, let $A_p = \{x\colon \varepsilon > 0 \text{ implies } \int_{x-\varepsilon}^{x+\varepsilon} |f(t)|^p \, dt = \infty\}$. Then A_p is clearly closed, and Lemma 11 of [5, p. 89] shows that if f is L_p-smooth then A_p is countable. Let $[0, 1] \setminus A_p = B_p$. It follows readily, as in the proof of [4, p. 90], that f is Baire* 1 on each component of B_p. Since A_p is closed and countable, it is also immediate that f is Baire* 1 on $[0, 1]$.

It should be noted that if a function f is smooth and measurable, then it is L_p-smooth. Therefore, Theorem 5 also applies to smooth measurable functions.

THEOREM 6. *Let* $f\colon [0, 1] \to R$ *be* L_p*-smooth and Darboux, and let* $S = \{x\colon L_p f' \text{ exists}\}$. *Then* $L_p f'$ *has the Darboux property on* S.

PROOF. It need only be shown that if $a < b$ and $L_p f'(a) < 0 < L_p f'(b)$, then there is a point x_0 in (a, b) with $L_p f'(x_0) = 0$. Consider f on the interval $[a, b]$. If f is continuous on $[a, b]$ then the result follows as in Theorem 3 of [5, p. 85]. If f is not continuous on $[a, b]$, then because it is Darboux it cannot be monotone. Hence it is possible to select two points a_1 and b_1 with $a \leqslant a_1 < b_1 \leqslant b$ and $f(a_1) = f(b_1)$. The function f is Baire* 1, Darboux on $[a_1, b_1]$. Hence Theorem 3 of this paper implies that there is a point x_0 in (a_1, b_1) at which f has a local maximum or local minimum. Then when h is sufficiently small it follows that

$$|f(x_0 + t) - f(x_0)|^p \leqslant |f(x_0 + t) + f(x_0 - t) - 2f(x_0)|^p \quad \text{for all } |t| < h.$$

Hence f has an L_p-derivative of 0 at x_0. Finally, it should be remarked that if f is smooth and measurable then $S = \{x\colon L_p f' \text{ exists}\}$ can be replaced by $\{x\colon f' \text{ exists}\}$.

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