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A NOTE ON RIESZ OPERATORS

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ABSTRACT. The purpose of this note is to settle a problem posed by Caradus, Pfaffenberger, and Yood; namely, it is proved that every Riesz operator R on a Hilbert space has a decomposition $R = C + Q$ where C is compact and both Q and $CQ - QC$ are quasinilpotent.

Let H denote a complex, separable, infinite dimensional Hilbert space. In [5], West showed that every Riesz operator was a decomposable Riesz operator, i.e., $R = C + Q$ where C is a compact operator and Q is quasinilpotent. In general, this decomposition is not unique.

A Riesz operator is said to be fully decomposable if R is decomposable and, in addition, C commutes with Q for some decomposition C and Q.

In [1, p. 58], an example of Gillespie and West was given showing that there are some Riesz operators on H which are not fully decomposable. They produced a Riesz operator R for which no decomposition could commute. This leads to the following question proposed in [1, p. 59]: Can every Riesz operator be decomposed in such a manner that the commutator $CQ - QC$ is quasinilpotent? The purpose of this note is to give a positive answer to this question, and, in fact, a slightly stronger result is proved.

The key to our proof is a lemma of Gohberg and Krein which was stated without proof in [3, p. 17] and was later stated and proved by Stampfli [4].

LEMMA 1 [GOHBERG-KREIN, STAMPFLI]. *Let E be a closed set in* $\mathbf{C}$. *Let $\sigma(T) \setminus E$ consist of isolated points $\{\lambda_j\}$ which of necessity cluster only on E. Let each λ_j be a point of finite multiplicity. Then, $T = S + K$ where K is compact and $\sigma(S) \subset E$.*

Our theorem will depend heavily on the "Stampfli decomposition" and on its notation. Let us recall the pertinent steps. It was shown by Stampfli [4] that for a T satisfying the hypotheses of Lemma 1,

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$$T = \left(\begin{array}{ccccc|c} \lambda_1 & \ddots & & & & \\ & & \lambda_1 & & * & * \\ & 0 & & \lambda_2 & \ddots & \\ & & & & \lambda_2 \ \ddots & \\ \hline & & & 0 & & L \end{array}\right)$$

That is, if

$$P_j = \frac{1}{2\pi i} \int_{|\lambda - \lambda_j| = \varepsilon_j} (\lambda - T)^{-1} \, d\lambda$$

where $0 < \varepsilon_j < \min\{\min_{i \neq j}(|\lambda_i - \lambda_j|), \operatorname{dist}(\lambda_j, E)\}$ for $i \neq j$, then T has the matrix form listed above, where $L = QTQ$ and Q is the orthogonal projection on $(\Sigma P_j H)^{\perp}$. Now let $\{\alpha_k\}$ be a countable dense subset of E. With each λ_j, associate an α_{k_j} as follows. Choose α_{k_j} such that $|\alpha_{k_j} - \lambda_j| \leqslant 2 \operatorname{dist}(\lambda_j, E)$. For simplicity write α_{k_j} as α_j. Next set

$$K = \left(\begin{array}{ccccc|c} \lambda_1 - \alpha_1 & \ddots & & & & \\ & & \lambda_1 - \alpha_1 & 0 & & \\ & & \lambda_2 - \alpha_2 & \ddots & & 0 \\ & 0 & & & \lambda_2 - \alpha_2 \ \ddots & \\ \hline & & 0 & & & 0 \end{array}\right)$$

and define $S = T - K$. Obviously, K is compact.

We are now ready to state our theorem.

THEOREM 1. *Let T satisfy the hypotheses of Lemma* 1. *Then in the "Stampfli decomposition" $T = S + K$, the commutator $SK - KS$ is a compact quasi-nilpotent operator.*

PROOF. Clearly $SK - KS$ is compact. Using the above notation, it is easily seen that

$$SK - KS = \begin{pmatrix} \hat{S}\hat{K} - \hat{K}\hat{S} & \hat{K}^* \\ 0 & 0 \end{pmatrix}$$

where $\hat{K}^*$ denotes the product of $\hat{K}$ and the northeast block of S and $\hat{S}$ denotes the northwest corner of S.

We first show that $\hat{S}\hat{K} - \hat{K}\hat{S}$, viewed as an operator on $\Sigma \overline{P_j H}$ is quasi-nilpotent. As a matrix $\hat{S}\hat{K} - \hat{K}\hat{S}$ is a compact operator, upper triangular with main diagonal identically zero. Let $K_n = P_n(\hat{S}\hat{K} - \hat{K}\hat{S})P_n$ where P_n projects onto $\mathrm{sp}\{e_1, \ldots, e_n\}$ where $\{e_i\}_{i=1}^{\infty}$ is the orthonormal basis for which $\hat{S}$ is upper triangular. Since P_n converges to the identity in the strong operator topology, K_n converges uniformly to $\hat{S}\hat{K} - \hat{K}\hat{S}$. Clearly, each K_n is quasinilpotent (actually nilpotent), so by [3, Theorem 4.1], $\hat{S}\hat{K} - \hat{K}\hat{S}$, as a uniform limit of compact quasinilpotent operators, is quasinilpotent.

To complete the proof, it suffices to show that $SK - KS$ is quasinilpotent. By the Riesz spectral theorem for compact operators, this is equivalent to showing that $SK - KS$ has no nonzero eigenvalues.

So assume $\lambda \neq 0$ and

$$\begin{pmatrix} \hat{S}\hat{K} - \hat{K}\hat{S} & \hat{K}^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \lambda\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Upon equating components of the vectors, we see that

$$e_2 = 0 \quad \text{and} \quad (\hat{S}\hat{K} - \hat{K}\hat{S})e_1 = \lambda e_1$$

which is impossible; thus $SK - KS$ is quasinilpotent. This completes the proof.

COROLLARY. *For a Riesz operator R on a Hilbert space, we have $R = C + Q$ where C is compact and both Q and $CQ - QC$ are quasinilpotent.*

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