PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

REDUCTION OF SYSTEMS OF LINEAR EQUATIONS IN ORDINAL VARIABLES

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ABSTRACT. In this note we are concerned with a general finite system

$$\text{(S)} \qquad \sum_{i=0}^{n-1} x_i \alpha_{ji} = \beta_j; \qquad j < m,$$

of m linear equations in n variables, where the α_{ji} and the β_j are positive ordinals, and the variables x_i range over ordinals.

In the particular case $n = 1$ we show that (S) can be reduced to a canonical form (S^*) having solutions of a relatively simple type, and we use (S^*) to obtain the solution-set of (S).

In the general case we show that (S) can be reduced to a finite sequence of single-variable systems, and again obtain the solution-set of (S) in terms of the solution-sets of these simpler systems.

We assume a knowledge of the elementary theory of ordinal arithmetic, such as may be found for example in [2].

Ordinals will generally be denoted by lower-case Greek letters, with finite ordinals (natural numbers) being denoted by lower-case Latin letters; the first transfinite ordinal will always be denoted by "ω".

For $\alpha > 0$, let $\alpha = \sum_{i=0}^{n} \omega^{e_i(\alpha)} c_i(\alpha)$ be the (Cantor) normal form of α. We put $l(\alpha) = n + 1$ (the "length" of α), $e(\alpha) = e_0(\alpha)$ (the "degree" of α), $c(\alpha) = c_n(\alpha)$, and $r(\alpha) = \omega^{e_n(\alpha)}$. Of course $r(\alpha)$ is the smallest positive remainder of α, and α is a successor if and only if $r(\alpha) = 1$. Finally, if $r(\alpha) = 1$, then we put $I(\alpha) = \sum_{i=0}^{n-1} \omega^{e_i(\alpha)} c_i(\alpha)$.

We note that for all $\alpha, \beta > 0$ we have either $l(\alpha\beta) = l(\beta)$ or else $l(\alpha\beta) = l(\alpha) + l(\beta) - 1$, and that $l(\alpha\beta) = l(\beta)$ if and only if either $l(\alpha) = 1$ or $r(\beta) > 1$. These facts can easily be verified by expanding α and β into normal form and multiplying out.

We require the following results on right-divisors of ordinals; these are (in essence) set forth in [1].

RESULT 1. A nonzero limit ordinal α is a right-divisor of a nonzero (limit) ordinal β if and only if the following hold:

(a) $l(\alpha) = l(\beta)$;

(b) There is some $\delta < e(r(\beta))$ such that $\delta + e_i(\alpha) = e_i(\beta)$ and $c_i(\alpha) = c_i(\beta)$ for every $i < l(\alpha)$.

In connection with the above, we note that if α is a nonzero limit ordinal and δ is any ordinal, then $\psi\alpha = \omega^\delta\alpha$ for any ordinal ψ with $\omega^\delta \leqslant \psi < \omega^{\delta+1}$.

Received by the editors November 19, 1974 and, in revised form, July 24, 1975.

AMS (MOS) subject classifications (1970). Primary 04A10.

[1] The work contained in this paper was done whilst the author held a Research Fellowship at the Australian National University.

For we have $\psi = \omega^\delta c + \gamma$ for some positive number c and some ordinal $\gamma < \omega^\delta$, and we also have $\alpha = \omega\theta$ for some positive ordinal θ. But then $\psi\alpha = ((\omega^\delta c + \gamma)\omega)\theta = ((\omega^\delta c)\omega)\theta = \omega^\delta\omega\theta = \omega^\delta\alpha$.

RESULT 2. Let α be a successor ordinal of length $p + 2$. Then α is a right-divisor of a nonzero ordinal β if and only if the following hold:

(a) $l(\beta) \geqslant p + 2$;
(b) $c(\alpha)$ divides $c_{p+1}(\beta)$;
(c) $\omega^{e_{p+1}(\beta)}I(\alpha) = \sum_{i=0}^{p} \omega^{e_i(\beta)}c_i(\beta)$.

In this case the equation $x\alpha = \beta$ has a unique solution.

Consider the system

(S1) $$x\alpha_j = \beta_j; \qquad j < m,$$

of m linear equations in 1 variable. Obviously if (S1) has a solution, then we have

(C1) $$\alpha_j \text{ right-divides } \beta_j; \qquad j < m.$$

THEOREM 1. *Let the system*

(S1) $$x\alpha_j = \beta_j; \qquad j < m,$$

of m linear equations in 1 *variable satisfy condition* (C1). *Then there is a system*

(S1*) $$y\alpha_j = \tau_j; \qquad j < m,$$

such that:

(1) $l(\alpha_j) = l(\tau_j)$ *for every* $j < m$;
(2) *If* (S1*) *has a solution, then it has a solution* $y = \omega^\delta$ *for some* δ;
(3) (S1) *has a solution if and only if the following hold*:
  (I) (S1*) *has a solution*;
  (II) *For all* $i, j < m$ *such that* $r(\alpha_i) = r(\alpha_j) = 1$ *we have*

(IIa) $$c(\alpha_i)c_{q_j}(\beta_j) = c(\alpha_j)c_{q_i}(\beta_i),$$

where $q_i = l(\alpha_i) - 1$ *and* $q_j = l(\alpha_j) - 1$;

(IIb) $$\sum \{\omega^{e_k(\beta_i)}c_k(\beta_i); l(\alpha_i) \leqslant k < l(\beta_i)\} = \sum \{\omega^{e_k(\beta_j)}c_k(\beta_j); l(\alpha_j) \leqslant k < l(\beta_j)\}.$$

PROOF. We define the τ_j as follows. If $r(\alpha_j) > 1$, put $\tau_j = \beta_j$. Otherwise, put

$$\tau_j = \left(\sum_{i=0}^{p-1} \omega^{e_i(\beta_j)}c_i(\beta_j)\right) + \omega^{e_p(\beta_j)}c(\alpha_j),$$

where $p = l(\alpha_j) - 1$.

This defines (S1*): since (S1) satisfies (C1), it follows from Result 1 that (S1*) satisfies condition (1).

We now consider two cases.

(A) $r(\alpha_j) > 1$ for every $j < m$. From the above definition of the τ_j we see that (S1*) is simply (S1), and from this and our assumption on the $r(\alpha_j)$, condition (3) follows trivially. It therefore suffices to demonstrate (2), and so we assume that (S1*) has a solution, $y = \psi$. Now there is an ordinal δ such that $\omega^\delta \leqslant \psi < \omega^{\delta+1}$; since $r(\alpha_j) > 1$ for each j, the remark following Result 1 tells us that $\psi\alpha_j = \omega^\delta\alpha_j$ for each j. Thus $y = \omega^\delta$ is also a solution.

(B) $r(\alpha_j) = 1$ *for some* $j < m$.

We commence by proving (2); thus let $y = \psi$ be a solution of (S1*). From the remark following Result 2, we see that $y = \psi$ is the only solution of (S1*), and so we must show that $\psi = \omega^\delta$ for some δ.

Choose $j^\circ < m$ such that $r(\alpha_{j^\circ}) = 1$. From (1) we have $l(\alpha_{j^\circ}) = l(\tau_{j^\circ}) = l(\psi\alpha_{j^\circ})$, and so from the remark preceding Result 1 we have that $l(\psi) = 1$, that is, $\psi = \omega^\delta b$ for some δ and some nonzero number b. However, we also have $bc(\alpha_{j^\circ}) = c(\psi\alpha_{j^\circ}) = c_p(\tau_{j^\circ}) = c(\alpha_{j^\circ})$, where $p = l(\alpha_{j^\circ}) - 1$. Hence $b = 1$, and so $\psi = \omega^\delta$ as desired.

It remains to demonstrate condition (3). Thus suppose that $x = \psi$ is a solution of (S1), and let δ be such that $\omega^\delta \leqslant \psi < \omega^{\delta+1}$. If $r(\alpha_j) > 1$, then as before we have $\omega^\delta\alpha_j = \psi\alpha_j = \beta_j = \tau_j$. If, on the other hand, $r(\alpha_j) = 1$, then we have $\omega^\delta\alpha_j = \omega^\delta I(\alpha_j) + \omega^\delta c(\alpha_j)$: we must show that the right side of this equation is in fact τ_j. Put $\psi = \omega^\delta b + \gamma$ for some nonzero number b and some $\gamma < \omega^\delta$; then $\beta_j = \psi\alpha_j = \omega^\delta I(\alpha_j) + \omega^\delta bc(\alpha_j) + \gamma$. From this it is easily seen that if we set $p = l(\alpha_j) - 1$, then

$$\delta = e_p(\beta_j) \quad \text{and} \quad \omega^\delta I(\alpha_j) = \sum_{i=0}^{p-1} \omega^{e_i(\beta_j)} c_i(\beta_j).$$

Hence we do indeed have $\omega^\delta\alpha_j = \tau_j$. Thus $y = \omega^\delta$ is a solution of (S1*), and so (I) holds.

With $x = \psi$ as above, we now have to demonstrate (II), and so we take $i, j < m$ and assume that $r(\alpha_i) = r(\alpha_j) = 1$. As above we have $\beta_j = \omega^\delta I(\alpha_j) + \omega^\delta bc(\alpha_j) + \gamma$, with of course a similar expression for β_i. Thus β_i, β_j have the common remainder γ, and since $l(\gamma) = l(\beta_j) - l(\alpha_j) = l(\beta_i) - l(\alpha_i)$, this establishes (IIb). But the same expressions for β_i and β_j tell us that $c_{q_i}(\beta_i)/c(\alpha_i) = b = c_{q_j}(\beta_j)/c(\alpha_j)$, with q_i, q_j as in (IIa). This establishes (IIa).

Suppose now that (I) and (II) hold, and let $y = \sigma$ be a solution of (S1*). From our assumption that $r(\alpha_j) = 1$ for some $j < m$ it follows that $y = \sigma$ is the only solution of (S1*), and so from (2), which has already been established, we conclude that $\sigma = \omega^\delta$ for some δ.

Take a particular $i < m$ for which $r(\alpha_i) = 1$, and put $p = l(\alpha_i) - 1$, $q = l(\beta_i) - 1$. Now define a number b and an ordinal γ by $b = c_p(\beta_i)/c(\alpha_i)$ and $\gamma = \sum \{\omega^{e_k(\beta_i)} c_k(\beta_i); p < k \leqslant q\}$. By assumption, (S1) satisfies (C1); thus by Result 2 b is well defined and is positive. Furthermore, (II) tells us that b and γ are independent of the particular choice of i.

Now for any such i we have $\sigma\alpha_i = \tau_i$, whence from the definition of τ_i we see that $\delta = e(r(\tau_i)) = e_p(\beta_i)$, p as above. Hence $\gamma < \sigma$, and we now put $\psi = \sigma b + \gamma$. We claim that $x = \psi$ is a solution of (S1). For if $r(\alpha_j) > 1$, we have $\psi\alpha_j = \omega^\delta\alpha_j = \tau_j = \beta_j$, since $\omega^\delta \leqslant \psi < \omega^{\delta+1}$. On the other hand, if $r(\alpha_j) = 1$, then $\psi\alpha_j = \omega^\delta I(\alpha_j) + \omega^\delta bc(\alpha_j) + \gamma = \beta_j$. This proves our theorem.

COROLLARY. *Assume that the system*

(S1) $$x\alpha_j = \beta_j; \qquad j < m,$$

with α_j, β_j positive ordinals, has a solution. Then:

(1) *If $r(\alpha_j) > 1$ for every $j < m$, then $x = \psi$ is a solution of* (S1) *if and only if $\omega^\delta \leqslant \psi < \omega^{\delta+1}$ for some δ such that $\delta + e(\alpha_j) = e(\beta_j)$ for every $j < m$.*

(2) *If $r(\alpha_i) = 1$ for some $i < m$, then* (S1) *has the unique solution*

$$x = \omega^{e_p(\beta_i)}[c_p(\beta_i)/c(\alpha_i)] + \sum \{\omega^{e_k(\beta_i)}c_k(\beta_i); p < k \leqslant q\},$$

where $p = l(\alpha_i) - 1$, $q = l(\beta_i) - 1$.

THEOREM 2. *The system*

(S) $$\sum_{i=0}^{n-1} x_i\alpha_{ij} = \beta_j; \qquad j < m,$$

with α_{ij}, β_j positive ordinals, has a solution if and only if there are numbers $i_0 < i_1 < \cdots < i_s < n$ and ordinals ρ_{i_kj}, $k \leqslant s$, with $\beta_j = \sum_{k=0}^{s} \rho_{i_kj}$ for every $j < m$, such that the $s + 1$ systems

(S_k) $$x_{i_k}\alpha_{i_kj} = \rho_{i_kj}; \qquad j < m, k \leqslant s,$$

all possess solutions.

PROOF. The sufficiency of the condition is clear, since if each (S_k) has a solution $x_{i_k} = \psi_k$, then a solution of (S) is given by $x_i = \psi_k$ if $i = i_k$ for some $k \leqslant s$ and $x_i = 0$ otherwise.

The necessity is proved by induction on n, the case $n = 1$ being trivial. Thus take $t > 1$ and assume that the necessity has been proved for every $n < t$. Suppose that the system (S) with $n = t$ has a solution $x_i = \gamma_i$, $i < t$. As each β_j is positive, we must have $\gamma_i > 0$ for some $i < t$. Let $i°$ be the largest such i, and for each $j < m$ define ρ_j, τ_j by $\rho_j = \gamma_{i°}\alpha_{i°j}$ and $\tau_j = \sum_{i=0}^{i°-1} \gamma_i\alpha_{ij}$. Then $\beta_j = \tau_j + \rho_j$, and the two systems

(S*) $$\sum_{i=0}^{i°-1} x_i\alpha_{ij} = \tau_j; \qquad j < m,$$

($S^\#$) $$x_{i°}\alpha_{i°j} = \rho_j; \qquad j < m,$$

possess solutions.

As $i° < t$ we can apply the induction hypothesis to (S*), and the desired result is immediate.

Let us call a sequence of systems

(S_k) $$x_{i_k}\alpha_{i_kj} = \rho_{i_kj}; \qquad j < m, k \leqslant s$$

"compatible" with a system

(S) $$\sum_{i=0}^{n-1} x_i\alpha_{ij} = \beta_j; \qquad j < m,$$

if $i_0 < i_1 < \cdots < i_s < n$ and $\beta_j = \sum_{k=0}^{s} \rho_{i_kj}$ for every $j < m$, with each ρ_{i_kj} positive.

Clearly every solution of (S) determines a compatible sequence, but it is also clear that a compatible sequence does not necessarily determine a unique solution of (S).

The proof of the following result is straightforward.

THEOREM 3. *Assume that the system* (S) (*as above*) *has a solution. Then* $x_i = \gamma_i$ *is a solution of* (S) *if and only if some compatible sequence has a solution* $x_{i_k} = \psi_k$, $k \leqslant s$, *such that*

(1) $\gamma_{i_k} = \psi_k$ *for every* $k \leqslant s$;

(2) $\gamma_i = 0$ *for every* i *with* $i_s < i < n$;

(3) $\gamma_i \alpha_{ij} \omega \leqslant \rho_{i_{k+1}j}$ *for every* $j < m$ *and every* i *with* $i_k < i < i_{k+1}$ *for some* $k < s$.

The author wishes to thank the referee for his comments.

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