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A CHARACTERIZATION OF μ-SEMIRINGS

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ABSTRACT. A characterization of μ-semirings is given, namely, "A semiring $\mathfrak{S}$ is a μ-semiring, if and only if, for each ideal $\mathfrak{a}$ of $\mathfrak{S}$ with no subideals in a π-system $\mathfrak{P}$, there exists a maximal ideal which has no subideals in $\mathfrak{P}$ and contains $\mathfrak{a}$."

1. **Introduction**. A *semiring* is an algebraic system $\mathfrak{S} = \{a, b, c, \dots\}$ in which two binary associative operations, called sum $(+)$ and product $(\cdot)$, are defined so that the operation $\cdot$ is both left- and right-distributive over $+$. A subset $\mathfrak{a}$ of $\mathfrak{S}$ is called an ideal if: (i) $a, b \in \mathfrak{a}$ imply $a + b \in \mathfrak{a}$; (ii) $a \in \mathfrak{a}$, $s \in \mathfrak{S}$ imply $as \in \mathfrak{a}$, $sa \in \mathfrak{a}$.

A subset M of $\mathfrak{S}$ is called an *m-system* of $\mathfrak{S}$ if, for each pair $a, b \in M$, there exists $x \in \mathfrak{S}$ such that $axb \in M$; a subset P of $\mathfrak{S}$ is called a *p-system* of $\mathfrak{S}$ if, for each $a \in P$, there exists $x \in P$ such that $axa \in P$. These concepts, stemming from ring theory, allow us, as in that theory, to make the study of prime and semiprime ideals and to introduce the notion of the Baer-McCoy-Levitzki radical [1].

Lattice semirings are instances of interesting semirings. $\mathfrak{S}$ is a *lattice semiring* if: (i) $\mathfrak{S}$ is a lattice besides being a semiring; (ii) the operations $\wedge$, $\vee$ satisfy $x + y = x \vee y$, $xy \leqslant x \wedge y$. For these semirings, M. L. Noronha Galvão gave [5] a theory for primary and primal ideals analogous to the theory of Noether-Krull-Fuchs.

Important examples of lattice semirings are the sets $\bar{\mathfrak{S}}$ of all ideals either of a ring or of a semiring or of a semigroup. m-systems and p-systems of $\bar{\mathfrak{S}}$ are called by A. Almeida Costa [2], respectively, μ-systems and π-systems of $\mathfrak{S}$. Consequently, leaving aside $\bar{\mathfrak{S}}$, a set $\mathfrak{M}$ of ideals of a semiring $\mathfrak{S}$ is a μ-system, if and only if, for each pair $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$, there exists an ideal $\mathfrak{x}$ of $\mathfrak{S}$ such that $\mathfrak{a}\mathfrak{x}\mathfrak{b} \in \mathfrak{M}$; a set $\mathfrak{P}$ of ideals of $\mathfrak{S}$ is a π-system if and only if, for each $\mathfrak{a} \in \mathfrak{P}$, there exists an ideal $\mathfrak{x}$ of $\mathfrak{S}$ such that $\mathfrak{a}\mathfrak{x}\mathfrak{a} \in \mathfrak{P}$. Moreover, in any semiring $\mathfrak{S}$ the set of all ideals which are not contained in a given prime ideal is a μ-system and the set of all ideals which are not contained in a given semiprime ideal is a π-system.

A *μ-semiring* is a semiring which satisfies either the μ-condition or the

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π-condition. These conditions are defined as follows (we denote by $C(\mathfrak{x})$ the set of all ideals which are not subideals of $\mathfrak{x}$):

(μ) For every μ-system $\mathfrak{M}$ and every chain of ideals $\{\mathfrak{a}_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathfrak{M} \subseteq C(\mathfrak{a}_\lambda)$ ($\lambda \in \Lambda$) one has $\mathfrak{M} \subseteq C(\bigcup \mathfrak{a}_\lambda)$;

(π) For every π-system $\mathfrak{P}$ and every chain of ideals $\{\mathfrak{a}_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathfrak{P} \subseteq C(\mathfrak{a}_\lambda)$ ($\lambda \in \Lambda$) one has $\mathfrak{P} \subseteq C(\bigcup \mathfrak{a}_\lambda)$.

These assertions are equivalent, as proved in [4] and [5] where the theory of μ-semirings is developed. These assertions are also equivalent to the following:

(μ_1) For every μ-system $\mathfrak{M}$ and every chain of ideals $\{\mathfrak{a}_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathfrak{M} \subseteq C(\mathfrak{a}_\lambda)$ ($\lambda \in \Lambda$) there is an ideal $\mathfrak{a}$ such that $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ ($\lambda \in \Lambda$), $\mathfrak{M} \subseteq C(\mathfrak{a})$;

(π_1) For every π-system $\mathfrak{P}$ and every chain of ideals $\{\mathfrak{a}_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathfrak{P} \subseteq C(\mathfrak{a}_\lambda)$ ($\lambda \in \Lambda$) there is an ideal $\mathfrak{a}$ such that $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ ($\lambda \in \Lambda$), $\mathfrak{P} \subseteq C(\mathfrak{a})$.

Noetherian semirings, that is, those which satisfy the a.c.c. for ideals (in particular, semirings of finite order) and non-Noetherian semirings consisting of the real numbers $x \geqslant r$, where $r > 1$ is a real number [3], provide examples of μ-semirings.

In the general theory of semirings the use of certain μ-systems and certain π-systems (said "particulars") has permitted the establishment of results concerning prime and semiprime ideals and consequent radical theories, but in the theory of μ-semirings the use of μ-systems and π-systems is sufficient to establish the Noether-Krull-Fuchs results.

Let us take in a μ-semiring a μ-system $\mathfrak{M}$ (π-system $\mathfrak{P}$) and an ideal $\mathfrak{a}$ with no subideals in $\mathfrak{M}$ (in $\mathfrak{P}$). From Zorn's lemma it follows that there is a maximal ideal which has no subideals in $\mathfrak{M}$ ($\mathfrak{P}$) and contains $\mathfrak{a}$.

In this note we will prove the following characterization of μ-semirings:

A semiring $\mathfrak{S}$ is a μ-semiring if and only if it satisfies the condition:

(π_0) *For each ideal $\mathfrak{a}$ and for each π-system $\mathfrak{P}$ such that $\mathfrak{a}$ has no subideals in $\mathfrak{P}$, i.e., $\mathfrak{P} \subseteq C(\mathfrak{a})$, there exists a maximal ideal $\mathfrak{y}$ which has no subideals in $\mathfrak{P}$ and contains $\mathfrak{a}$, i.e., $\mathfrak{P} \subseteq C(\mathfrak{y}) \subseteq C(\mathfrak{a})$.*

2. **Preliminary propositions**. We first prove:

PROPOSITION 1. *Let $\mathfrak{P}$ be a π-system. If there is a maximal ideal $\mathfrak{y}$ with no subideals in $\mathfrak{P}$, i.e., $\mathfrak{P} \subseteq C(\mathfrak{y})$, then $\mathfrak{y}$ is a semiprime ideal.*

PROOF. Let us assume that $\mathfrak{y}$ is not semiprime. Then for an ideal $\mathfrak{x}$ one has $\mathfrak{x}^2 \subseteq \mathfrak{y}$, $\mathfrak{x} \nsubseteq \mathfrak{y}$. Hence $\mathfrak{y} \subset (\mathfrak{x}, \mathfrak{y})$, the least ideal containing both $\mathfrak{x}$ and $\mathfrak{y}$. Since $\mathfrak{y}$ is maximal and has no subideals in $\mathfrak{P}$, there exists $\mathfrak{m} \in \mathfrak{P}$ such that $\mathfrak{m} \subseteq (\mathfrak{x}, \mathfrak{y})$. Let us consider an ideal $\mathfrak{z}$ such that $\mathfrak{m}\mathfrak{z}\mathfrak{m} \in \mathfrak{P}$. The inclusions $\mathfrak{m}\mathfrak{z}\mathfrak{m} \subseteq \mathfrak{m}^2 \subseteq (\mathfrak{x}, \mathfrak{m})(\mathfrak{x}, \mathfrak{m}) \subseteq \mathfrak{y}$ contradict the hypothesis about $\mathfrak{y}$. Hence $\mathfrak{x}^2 \subseteq \mathfrak{y}$ implies $\mathfrak{x} \subseteq \mathfrak{y}$.

Let $\mathfrak{P}$ be a π-system and $\mathfrak{a}$ an ideal such that $\mathfrak{P} \subseteq C(\mathfrak{a})$; then a maximal ideal $\mathfrak{y}$ such that $\mathfrak{P} \subseteq C(\mathfrak{y}) \subseteq C(\mathfrak{a})$ is, of course, a maximal ideal satisfying $\mathfrak{P} \subseteq C(\mathfrak{y})$. We have:

COROLLARY 1. *Let* $\mathfrak{P}$ *be a* π*-system and* $\mathfrak{a}$ *an ideal such that* $C(\mathfrak{a})$. *If there is a maximal ideal* $\mathfrak{y}$ *such that* $\mathfrak{P} \subseteq C(\mathfrak{y}) \subseteq C(\mathfrak{a})$, *then* $\mathfrak{y}$ *is a semiprime ideal.*

LEMMA 1. *Let* $\mathfrak{a}$ *be a semiring satisfying condition* (π_0). *Given a* π*-system* $\mathfrak{P}$ *and an ideal* $\mathfrak{a}$ *such that* $\mathfrak{P} \subseteq C(\mathfrak{a})$, *then there exists a minimal semiprime ideal* $\mathfrak{s}$ *such that* $\mathfrak{P} \subseteq C(\mathfrak{s}) \subseteq C(\mathfrak{a})$.

PROOF. Condition (π_0) implies the existence of a maximal ideal $\mathfrak{y}$ such that $\mathfrak{P} \subseteq C(\mathfrak{y}) \subseteq C(\mathfrak{a})$. Since, by Corollary 1, $\mathfrak{y}$ is semiprime, the intersection of all semiprime ideals $\mathfrak{x}$ such that $\mathfrak{P} \subseteq C(\mathfrak{x}) \subseteq C(\mathfrak{a})$ is the minimal semiprime ideal $\mathfrak{s}$ we are looking for.

Now, let $\mathcal{G}$ be a family of ideals of a semiring $\mathfrak{S}$ satisfying the following conditions: (G_1) $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{G}$ imply $(\mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{G}$; (G_2) $\mathfrak{g} \subseteq \mathfrak{g}_1 \in \mathcal{G}$ imply $\mathfrak{g} \in \mathcal{G}$ ($\mathcal{G}$ is an ideal of the lattice $\overline{\mathfrak{S}}$ of all ideals of $\mathfrak{S}$). It is easy to verify that the existence of a maximal element $\mathfrak{g}_0 \in \mathcal{G}$ implies $\mathfrak{g}_0 = \bigcup \mathfrak{g}_\alpha$ $(\mathfrak{g}_\alpha \in \mathcal{G})$. It is the same to say that $\mathfrak{g}_0$ is maximal in $\mathcal{G}$ or to say that $\mathfrak{g}_0$ is maximal such that $\overline{\mathfrak{S}} - \mathcal{G} = C(\mathfrak{g}_0)$.

LEMMA 2. *Let* $\mathfrak{S}$ *be a semiring satisfying condition* (π_0) *and let* $\{\mathfrak{s}_\lambda\}$ $(\lambda \in \Lambda)$ *be a chain of semiprime ideals; then* $\bigcup \mathfrak{s}_\lambda = \mathfrak{s}_{\lambda_0}$ *for some* $\lambda_0 \in \Lambda$.

PROOF. Let $\mathcal{G}$ be the family consisting of all subideals of all $\mathfrak{s}_\lambda$. $\mathcal{G}$ satisfies (G_1) and (G_2). We shall verify that the set of all ideals not in $\mathcal{G}$, $\mathfrak{P} = \overline{\mathfrak{S}} - \mathcal{G}$, is a π-system. Given $\mathfrak{x} \in \mathfrak{P}$ we shall prove that $\mathfrak{x}\mathfrak{x}^2\mathfrak{x} \in \mathfrak{P}$. If this were not so, one would have $\mathfrak{x}^2\mathfrak{x}^2 \in \mathcal{G}$, hence $\mathfrak{x}^2\mathfrak{x}^2 \subseteq \mathfrak{s}_\lambda$, for some $\lambda \in \Lambda$, which would imply $\mathfrak{x} \subseteq \mathfrak{s}_\lambda$, i.e., $\mathfrak{x} \in \mathcal{G}$, which is absurd. The fact that $\mathfrak{S}$ satisfies condition (π_0) and the inclusion $\mathfrak{P} \subseteq C(\mathfrak{s}_\lambda)$, together, imply the existence of a maximal ideal $\mathfrak{y}$ such that $\mathfrak{P} \subseteq C(\mathfrak{y})$. Thus we conclude the existence of a maximal ideal in $\mathcal{G}$, which is necessarily a $\mathfrak{s}_{\lambda_0}$ such that $\bigcup \mathfrak{s}_\lambda = \mathfrak{s}_{\lambda_0}$ $(\lambda \in \Lambda)$.

3. **Main proposition**. We have seen above, in the introduction, that the necessity of condition (π_0) for $\mathfrak{S}$ to be a μ-semiring is a consequence of Zorn's lemma. Conversely, let $\mathfrak{S}$ be a semiring that satisfies condition (π_0), let $\mathfrak{P}$ be a π-system, and let $\{\mathfrak{a}_\lambda\}$ $(\lambda \in \Lambda)$ be a chain of ideals of $\mathfrak{S}$ such that $\mathfrak{P} \subseteq C(\mathfrak{a}_\lambda)$ $(\lambda \in \Lambda)$. By Lemma 1, we can assign to each $\mathfrak{a}_\lambda$ the minimal semiprime ideal $\mathfrak{s}_\lambda$ such that $\mathfrak{P} \subseteq C(\mathfrak{s}_\lambda) \subseteq C(\mathfrak{a}_\lambda)$. From $\mathfrak{a}_\sigma \subseteq \mathfrak{a}_\tau$ one concludes $\mathfrak{P} \subseteq C(\mathfrak{s}_\tau) \subseteq C(\mathfrak{a}_\tau) \subseteq C(\mathfrak{a}_\sigma)$, hence by the minimality of $\mathfrak{s}_\sigma$, $\mathfrak{s}_\sigma \subseteq \mathfrak{s}_\tau$. Then, by Lemma 2 and by the fact that $\mathfrak{a}_\lambda \subseteq \mathfrak{s}_\lambda$, $\bigcup \mathfrak{a}_\lambda \subseteq \bigcup \mathfrak{s}_\lambda = \mathfrak{s}_{\lambda_0}$; consequently, $\mathfrak{P} \subseteq C(\mathfrak{s}_{\lambda_0}) \subseteq C(\bigcup \mathfrak{a}_\lambda)$. This completes the proof of the main proposition.

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