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A NOTE ON GREEN'S RELATIONS ON THE SEMIGROUP N_n

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ABSTRACT. Solvability criteria for nonnegative matrix equations are applied in characterizing the first three of the four Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$ on the semigroup N_n of all $n \times n$ nonnegative matrices. For the relation $\mathcal{J}$, it is shown that $\mathcal{D} = \mathcal{J}$ when the relation is restricted to the regular matrices in N_n although on the entire semigroup N_n, $n \geqslant 3$, $\mathcal{D} \neq \mathcal{J}$.

Introduction. In recent years, much research has been concerned with the development of the algebraic structure of the $n \times n$ nonnegative matrices. Topics investigated range from characterizing multiplicative groups [5] and semigroups [4], [11], [12], to initiating a theory of primes [1], [10]. A further topic which has received interest, in this regard, concerns the characterization of the Green's relations. In [6], the Green's relations on the semigroup Ω_n of the $n \times n$ doubly stochastic matrices were completely characterized. Similar characterizations were given in the semigroup $\mathcal{S}_n$ of $n \times n$ stochastic matrices. The study of the Green's relations on the semigroup N_n of $n \times n$ nonnegative matrices was begun in [7] and in [9]. The work herein is intended as a completion of that study.

As this work is a completion of previously published research, the paper will not contain a dictionary of its language. For this the reader is referred to [7].

The theory of Green's relations on N_n. To place the research in its proper framework, we will sketch the results of the theory up to the position held a priori this paper.

The work of [7] and [9] characterized the Green's relations on the set of regular elements in N_n. These characterizations are as follows.

THEOREM 1. *Let A and B be regular of rank r in N_n. Let P_1 and P_2 be permutation matrices such that $A_1 = AP_1$ and $B_1 = BP_2$ where A_1 and B_1 have the block form $[M \ U]$ where M is $n \times r$ and contains a monomial submatrix C of order r, i.e., $C = PD$ where D is a diagonal matrix with positive main diagonal and P a permutation matrix. Then $A \mathcal{R} B$ if and only if there exists a matrix $Q \in N_n$ of the form $Q = \left(\begin{smallmatrix} C & K \\ 0 & 0 \end{smallmatrix}\right)$ where C is $r \times r$ and monomial, such that $A_1 = B_1 Q$.*

We note that $A \mathcal{L} B$ in N_n if and only if $A^T \mathcal{R} B^T$ in N_n. Thus, our results are stated for the relation $\mathcal{R}$, the study of $\mathcal{L}$ being dual.

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THEOREM 2. *If A, B and N_n are regular then $A \mathcal{D} B$ if and only if they have the same rank.*

As a consequence of this theorem, the following corollary is obtained.

COROLLARY 1. *If A, B in N_n are regular, then $A \mathcal{D} B$ if and only if $A \mathcal{J} B$.*

PROOF. Since $\mathcal{D} \subseteq \mathcal{J}$ in any semigroup, $A \mathcal{D} B$ only if $A \mathcal{J} B$. Conversely, if $A \mathcal{J} B$ then the equations $A = X_1 B Y_1$ and $B = X_2 A Y_2$ are solvable for X_1, Y_1, and X_2, Y_2 in N_n. Thus, A and B have the same rank and hence from Theorem 2, $A \mathcal{D} B$.

From these theorems, it is seen that the tool used to characterize Green's relations for regular elements in N_n is that of rank. However, this tool is more of a vector space notion and as such is too sophisticated to characterize Green's relations on N_n. Here, a tool more concerned with positive cones, is necessitated. This work requires the following definitions.

Let $c(A)$ denote the cone generated by the columns of A. Define the cone dimension of $c(A)$, denoted $d(A)$, as the number of edges of $c(A)$. Further, as in [3], define a set T of column vectors in A to be independent if and only if each vector in T lies on an edge of $c(A)$ and no two vectors in T lie on the same edge of $c(A)$. A set of column vectors of A which is not independent is called dependent. Independent and dependent sets of row vectors in A are defined similarly.

For A, B in N_n, if A_i is a dependent column in A then $(BA)_i$ is a dependent column in BA. Hence $d(BA) \leqslant d(A)$. By utilizing this notion of cone dimension we now characterize the Green's relations on N_n. This characterization is founded on the following lemmas.

LEMMA 1. *Let A, B be in N_n.*
(i) *If $A \mathcal{R} B$ then $d(A) = d(B)$.*
(ii) *If $A \mathcal{D} B$ then $d(A) = d(B)$ and $d(A^T) = d(B^T)$.*

PROOF. Note that (i) follows from the definition of $\mathcal{R}$. For (ii), suppose $A \mathcal{R} C$ and $C \mathcal{L} B$. Then $d(A) = d(C)$ from (i). Since $C \mathcal{L} B$, $XC = B$ and $YB = C$ for some X, Y in N_n. But then, $d(B) \leqslant d(C)$ and $d(C) \leqslant d(B)$ and consequently $d(A) = d(B)$. Finally, as $A \mathcal{D} B$ if and only if $A^T \mathcal{D} B^T$, $d(A^T) = d(B^T)$.

LEMMA 2. *Let A be in N_n with $d(A) = c$. If A' is any $n \times c$ submatrix of independent columns of A then $A \mathcal{R} [A'\ 0]$. If further, $d(A^T) = r$ and A'' is any $r \times c$ submatrix in r independent rows and c independent columns of A then*

$$A \mathcal{D} \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$

PROOF. Without loss of generality suppose the c independent columns are in columns $1, \ldots, c$, i.e., $A = [A'\ A_2]$, where A' is $n \times c$. It is easily verified that $A \mathcal{R} [A'\ 0]$. If further, $d(A^T) = r$, then again without loss of generality, we assume the independent columns are in columns $1, \ldots, r$. Hence

$$A = \begin{pmatrix} A'' & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A'' is $r \times c$. As above,

$$A \,\mathcal{R} \begin{pmatrix} A'' & 0 \\ A_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A'' & 0 \\ A_3 & 0 \end{pmatrix} \mathcal{L} \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$A \,\mathcal{D} \begin{pmatrix} A'' & 0 \\ 0 & 0 \end{pmatrix}.$$

Based on these results, our characterization of the Green's relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{D}$ on N_n now follows.

THEOREM 3. *Let A, B be in N_n. The following statements are equivalent*:

(a) $A \,\mathcal{R}\, B$,

(b) (i) $d(A) = d(B) = d$ *and*

(ii) *given any $n \times d$ submatrix of independent columns of A, say A', and any $n \times d$ submatrix of independent columns of B say B', then there is a $d \times d$ monomial matrix X so that $A'X = B'$.*

PROOF. Suppose $d(A) = d$. Let A' be any submatrix of d independent columns of A. Now $A \,\mathcal{R}[A' \ 0]$ by Lemma 2. Similarly, if B' is any submatrix of d independent columns of B, then $B \,\mathcal{R}\, [B' \ 0]$.

Now if $A \mathcal{R} B$, then $d(A) = d(B)$ by Lemma 1. Further, from the above remarks, $A' \mathcal{R} B'$, i.e. $A'X = B'$ and $B'Y = A'$ hold for some X and Y in N_d. Hence $A'(XY) = A'$ and so $XY = I$ from which it follows that X and Y are monomials. Thus, (b) is obtained.

Conversely, if (b) holds, $A' \mathcal{R} B'$. Thus $[A' \ 0] \mathcal{R} [B' \ 0]$ where $[A' \ 0]$ and $[B' \ 0]$ are in N_n. As $A \mathcal{R} [A' \ 0]$ and $B \mathcal{R} [B' \ 0]$, (a) follows.

THEOREM 4. *Let A, B be in N_n. The following statements are equivalent*:

(a) $A \,\mathcal{D}\, B$.

(b) (i) $d(A) = d(B) = c$, $d(A^T) = d(B^T) = r$ *and*

(ii) *given any $r \times c$ submatrix A' in A and any $r \times c$ submatrix B' in B lying in r independent rows and c independent columns of A and B, respectively, then there are monomial matrices X in N_r and Y in N_c such that $XA'Y = B'$.*

PROOF. The argument is similar to that in Theorem 3.

Having characterized the Green's relations on N_n for $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{D}$, our efforts are now turned toward $\mathcal{J}$. Our work rests on the following corollary to Theorem 4.

COROLLARY. *Let A, B be in N_n and nonsingular. Then $A \,\mathcal{D}\, B$ if and only if $XAY = B$ has monomial solutions X and Y in N_n.*

Applying this corollary, we can now show that for $n \geqslant 3$, $\mathcal{D} \neq \mathcal{J}$ on N_n. For this consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 1 & 1 \end{pmatrix}.$$

Then, by direct calculation,

$$\begin{pmatrix}1&0&0\\0&1&0\\ \frac{21}{4}&0&\frac{1}{4}\end{pmatrix}\begin{pmatrix}1&0&0\\2&1&0\\3&4&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix} = \begin{pmatrix}1&0&0\\2&1&0\\6&1&1\end{pmatrix}$$

and

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0\\2&1&0\\6&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&7&2\end{pmatrix} = \begin{pmatrix}1&0&0\\2&1&0\\3&4&1\end{pmatrix}.$$

Hence $A\mathcal{J}B$. But, as there are no monomials D_1 and D_2 so that $D_1AD_2 = B$, it follows that $\mathcal{D} \neq \mathcal{J}$ on N_3.

For $n > 3$, consider

$$\bar{A} = \begin{pmatrix}A & 0\\ 0 & I_{n-3}\end{pmatrix} \quad \text{and} \quad \bar{B} = \begin{pmatrix}B & 0\\ 0 & I_{n-3}\end{pmatrix}.$$

From the above calculations, $\bar{A}\mathcal{J}\bar{B}$, yet $\bar{A}\not\mathcal{D}\bar{B}$. Hence $\mathcal{D} \neq \mathcal{J}$ on N_n, $n \geqslant 3$.

For $n = 2$, the result differs. For this case we show $\mathcal{D} = \mathcal{J}$. In this regard, suppose $A\mathcal{J}B$. We argue cases.

Case 1. A, and hence B, is singular.

Singularity here implies A and B are regular elements in N_2 and so $A\mathcal{D}B$.

Case 2. A, and hence B, is nonsingular.

By definition $A\mathcal{J}B$ implies that $X_1AY_1 = B$ and $X_2BY_2 = A$ for some nonsingular X_1, X_2, Y_1, and Y_2 in N_2. Thus, each of X_1, X_2, Y_1 and Y_2 has a positive diagonal. Let $X \prec Y$ denote the property that $x_{ij} > 0$ implies $y_{ij} > 0$ for all i, j. Then there exist permutation matrices P and Q so that $PAQ \prec B$ and permutation matrices R and S so that $RBS \prec A$. Thus, PAQ and B have the same 0 pattern. We again argue cases.

Case a. A, and hence B, has one or two zeros.

In this case, by solving equations, diagonal matrices D_1 and D_2 in N_2 may be found so that $D_1PAQD_2 = B$. Hence $A\mathcal{D}B$.

Case b. A, and hence B, is positive.

In this case, as $X_1AY_1 = B$ and $X_2BY_2 = A$ it follows that $(X_2X_1)A(Y_1Y_2) = A$. Set $X = X_2X_1$ and $Y = Y_1Y_2$, i.e. $XAY = A$. As $(cX)A(c^{-1}Y) = A$ for any positive number c, we may assume without loss of generality that $\det X = \det Y = \pm 1$. Suppose $\det X = \det Y = 1$, i.e. $x_{11}x_{22} - x_{12}x_{21} = 1$ and $y_{11}y_{22} - y_{12}y_{21} = 1$. Suppose

$$\max\{x_{11}, x_{22}\} = x_{11} \geqslant 1 \quad \text{and} \quad \max\{y_{11}, y_{22}\} = y_{11} \geqslant 1.$$

If either of these two inequalities is strict, the 1, 1 entry in XAY is strictly greater than a_{11}, a contradiction. But now $x_{11} = x_{22} = y_{11} = y_{22} = 1$. Further $x_{12} = x_{21} = y_{12} = y_{21} = 0$ so that $X = Y = I$. Considering all other possible cases leads to the conclusion that X and Y are monomials and so X_1, X_2, Y_1, and Y_2 are monomials, hence $A\ \mathcal{D}\ B$.

In conclusion, as $A\mathcal{J}B$ if and only if the equations $XAY = B$ and $XBY = A$ have solutions X_1, Y_1, X_2, Y_2 in N_n, respectively, and as $\mathcal{D} \neq \mathcal{J}$ on N_n for

$n \geqslant 3$, the authors suspect that no further satisfactory characterization of $\mathcal{J}$ exists. Thus, it is felt that the characterizations of the Green's relations on N_n are essentially completed by this work.

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