PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

THE JORDAN DECOMPOSITION OF VECTOR-VALUED MEASURES

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ABSTRACT. This paper gives criteria for a vector-valued Jordan decomposition theorem to hold. In particular, suppose L is an order complete vector lattice and $\mathcal{Q}$ is a Boolean algebra. Then an additive set function $\mu: \mathcal{Q} \to L$ can be expressed as the difference of two positive additive measures if and only if $\mu(\mathcal{Q})$ is order bounded. A sufficient condition for a countably additive set function with values in $c_0(\Gamma)$, for any set Γ, to be decomposed into difference of countably additive set functions is given; namely, the domain being the power set of some set.

We are concerned here with vector-valued additive set functions defined on some sort of Boolean algebra with vector-values in a (generally order complete) vector lattice. The purpose of this note is to expose conditions that insure such measures can be written as a difference of positive measures, i.e., conditions for a vector-valued Jordan decomposition theorem to hold. For this reason, a measure that can be expressed as the difference of two positive additive measures will be called "decomposable".

The decomposability of vector measures per se was first studied by C. E. Rickart in a 1943 Duke Mathematical Journal article where he established a Lebesgue decomposition theorem for the class of "strongly bounded" additive measures. This result was later re-established (although it was not realized at the time) by J. J. Uhl, Jr. [10] who also presented a Yosida-Hewitt decomposition theorem for "strongly bounded" measures. In [3], Diestel and Faires exhibited several decomposition theorems of the Jordan type, however, they did not give necessary and sufficient conditions for the decomposability of a vector measure. Our first result supplies these conditions, although, as is indicated, only one part of the proof is new.

So, let $\mathcal{Q}$ be a Boolean algebra, $S_{\mathcal{Q}}$ the vector lattice of simple functions over $\mathcal{Q}$ endowed with the uniform norm, and L a vector lattice. If $\mu: \mathcal{Q} \to L$ is an additive set function, then the operator $T_\mu: S_{\mathcal{Q}} \to L$ associated with μ is defined by

$$T_\mu\left(\sum_{i=1}^{n} \alpha_i c_{a_i}\right) = \sum_{i=1}^{n} \alpha_i \mu(a_i)$$

Received by the editors January 24, 1975 and, in revised form, February 12, 1976.
AMS (MOS) subject classifications (1970). Primary 46G10; Secondary 28A25.

where $\alpha_1, \ldots, \alpha_n$ are scalars and $a_1, \ldots, a_n$ are disjoint elements of $\mathcal{A}$. Furthermore, if we let $\Omega_{\mathcal{A}}$ denote the Stone space of $\mathcal{A}$, then by Stone's representation theorem we can identify $\mathcal{A}$ with the field of clopen subsets of $\Omega_{\mathcal{A}}$. We assume this identification has been made throughout the following.

LEMMA. *Let L be an order complete vector lattice and μ: $\mathcal{A} \to L$ an additive set function. If $\mu(\mathcal{A})$ is order bounded then the operator T_μ, associated with μ, maps order bounded sets in $S_{\mathcal{A}}$ into order bounded sets in L.*

PROOF. Since μ is order bounded there is a $z \in L^+$ such that $\mu(a) \subseteq \langle -z, z \rangle$ for all $a \in \mathcal{A}$. Let U denote the unit ball of $S_{\mathcal{A}}$ and note that any order interval in $S_{\mathcal{A}}$ is contained in an appropriate scalar multiple of U. Thus to establish the order boundedness of T_μ, it suffices to consider only simple functions of norm $\leqslant 1$. Let $\sum_{i=1}^n \alpha_i c_{a_i} \in U$ with $a_i \wedge a_j = 0$ for $i \neq j$, then $\sup_i |\alpha_i| \leqslant 1$ so, without loss of generality, we can assume the α_i's have the following order: $-1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \leqslant 1$. Letting $\beta_1 = \alpha_1, \beta_2 = \alpha_2 - \alpha_1, \ldots, \beta_n = \alpha_n - \alpha_{n-1}$ we have

$$T_\mu\left(\sum_{i=1}^n \alpha_i c_{a_i}\right) = \sum_{i=1}^n \alpha_i \mu(a_i)$$

$$= \beta_1 \mu\left(\bigvee_{i=1}^n a_i\right) + \beta_2 \mu\left(\bigvee_{i=2}^n a_i\right) + \cdots + \beta_{n-1} \mu\left(\bigvee_{i=n-1}^n a_i\right) + \beta_n \mu(a_n).$$

If we now let $b_k = \bigvee_{i=k}^n a_i$ for $k = 1, \ldots, n$, then

$$T_\mu\left(\sum_{i=1}^n \alpha_i c_{a_i}\right) = \sum_{k=1}^n \beta_k \mu(b_k).$$

Since $\sup_i |\alpha_i| \leqslant 1$, it is easily seen that $\sum_{k=1}^n |\beta_k| \leqslant 2$ so that

$$-2z \leqslant \sum_{k=1}^n \beta_k \mu(b_k) \leqslant 2z,$$

hence $-2z \leqslant T_\mu(\sum_{i=1}^n \alpha_i c_{a_i}) \leqslant 2z$ and T_μ is order bounded.

THEOREM 1. *Suppose L is an order complete vector lattice and $\mathcal{A}$ is a Boolean algebra. Then an additive set function μ: $\mathcal{A} \to L$ is decomposable if and only if $\mu(\mathcal{A})$ is order bounded.*

PROOF. Suppose μ is decomposable into $\mu = \mu^+ - \mu^-$. Then since $|\mu(a)| \leqslant \mu^+(1) + \mu^-(1)$ for every $a \in \mathcal{A}$, we clearly have

$$\mu(\mathcal{A}) \subseteq \langle -[\mu^+(1) + \mu^-(1)], [\mu^+(1) + \mu^-(1)] \rangle$$

so that $\mu(\mathcal{A})$ is order bounded.

On the other hand, if $\mu(\mathcal{A})$ is order bounded, by the Lemma T_μ: $S_{\mathcal{A}} \to L$ is order bounded. As is well known (see e.g. [8]), T_μ is then decomposable into $T_\mu = T_\mu^+ - T_\mu^-$ where T_μ^+, T_μ^- are positive linear operators. The proof is now complete upon the realization that by defining

$$\mu^+(a) = T_\mu^+(c_a) \quad \text{and} \quad \mu^-(a) = T_\mu^-(c_a)$$

for each $a \in \mathcal{C}$, we have the desired decomposition of μ.

The interest in this theorem comes somewhat less from its own statement than from the several corollaries which follow. It is worth noting at this time that we immediately get that if an additive set function μ takes values in any $C(K)$-space for K Stonian, then by the Dedekind complete nature of Stonian spaces, we have μ decomposable into the difference of positive measures. Somewhat less obvious is the following corollary due originally to A. Grothendieck and first exposed by Diestel and Faires [3]:

COROLLARY. *Let X be an abstract L-space and (Ω, Σ) a measure space. Then an additive map μ: $\Sigma \to X$ is decomposable if and only if μ is of bounded variation.*

Remarking that a Banach lattice containing no closed sublattice order isomorphic to c_0 has the monotone convergence property, Theorem 1 yields the

COROLLARY. *Let X be a Banach lattice containing no copy of c_0 and F be an X-valued measure defined on a sigma-algebra. Then F is decomposable if it is of finite variation.*

If we now look at arbitrary Banach lattice-valued measures of bounded variation, decomposability can still be obtained by considering a more restrictive class of set functions. In fact, a relatively straightforward calculation gives the following

PROPOSITION. *Let Ω be a set, Σ a sigma-algebra of subsets of Ω, and L a Banach lattice. Then a countably additive set function F: $\Sigma \to L$ possessing bounded variation $|F|$ with $dF/d|F|$ existing (in the sense of Bochner) is decomposable.*

REMARK. By using methods similar to those of J. J. Uhl, Jr. [9], one can obtain: If L is a Banach lattice, $\mathcal{F}$ a field of sets, and F: $\mathcal{F} \to L$ is a finitely additive measure possessing finite variation such that F is approximately differentiable with respect to $|F|$, then F is decomposable.

The next theorem shows when a countably additive set function with values in $c_0(\Gamma)$, for any set Γ, can be decomposed retaining countable additivity. More specifically,

THEOREM 2. *Let Ω be any set, denote by 2^Ω the power set of Ω, and let μ: $2^\Omega \to c_0(\Gamma)$ be countably additive. Then μ is decomposable into $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are both positive and countably additive.*

PROOF. Let $S = \{\sum_{i=1}^n \alpha_i c_{A_i}$: $\alpha_1, \ldots, \alpha_n$ are scalars and $A_1, \ldots, A_n$ is a sequence of pairwise disjoint members of $2^\Omega\}$ and define T_μ: $S \to c_0(\Gamma)$ by $T_\mu(s) = \sum_{i=1}^n \alpha_i \mu(A_i)$ where $s = \sum_{i=1}^n \alpha_i c_{A_i} \in S$. Note T_μ is continuous and linear on S viewed as a subspace of $l_\infty(\Omega)$. By the density of S in $l_\infty(\Omega)$, T_μ can be extended to a continuous operator $T: l_\infty(\Omega) \to c_0(\Gamma)$ where T is given by $T(f) = \int_\Omega f\,d\mu$ for each $f \in l_\infty(\Omega)$. As is well known [4], if λ is any localizable measure on a sigma-field S, and $F: S \to X$ is a bounded (finitely additive) vector measure vanishing on λ-null sets, F is countably additive if and only if the operator $T_F: l_\infty(\Omega) \to X$ given by $T_F(s) = \int s\,dF$ is weak*-weak continuous. Thus μ's countable additivity yields T is weak*-weak continuous and so its adjoint $T^*: l_1(\Gamma) \to l_1(\Omega)$ is a weakly compact linear operator. However, T^*'s range is contained in a Schur space so that T^*, and therefore T, is actually a compact operator. Hence, the image Λ of the unit ball of $l_\infty(\Omega)$, under the mapping T, is relatively compact. In particular, $\mu(2^\Omega) \subseteq \Lambda$ is relatively compact in $c_0(\Gamma)$. But in $c_0(\Gamma)$, relative compactness and order boundedness coincide, so by Theorem 1, μ is decomposable into $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are positive finitely additive set functions. Clearly both μ^+ and μ^- are strongly additive since $c_0(\Gamma)$ is weakly compactly generated.

Finally, it remains to show that μ^+ and μ^- can be taken to be countably additive. By the Uhl generalization of the Yosida-Hewitt decomposition theorem [10], there is a continuous linear norm one projection P of the strongly additive set functions onto the countably additive set functions. By a careful look at the construction involved in this theorem, one sees that the projection takes positive set functions to positive set functions, and so the desired decomposition of μ is complete upon the calculation $\mu = P\mu = P(\mu^+ - \mu^-) = P\mu^+ - P\mu^-$ where $P\mu^+$ and $P\mu^-$ are both positive and countably additive.

At this point it is worthwhile to return to Theorem 1 and note that decomposability of μ implying order boundedness of its range was not dependent on the order completeness of L. Thus, if μ is a decomposable additive set function from 2^Ω into any vector lattice L, then $\mu(2^\Omega)$ is order bounded. In particular, if $\mu: 2^\Omega \to c_0$ is decomposable, then its range must be relatively compact in c_0. Thus, as the following example demonstrates, Theorem 2 is highly dependent on the underlying sigma-algebra being all subsets of a given set.

EXAMPLE. Let $\Omega = [0, 2\pi]$, Σ be the collection of Lebesgue measurable subsets of Ω, and define $\mu: \Sigma \to c_0$ by

$$\mu(A) = \left(\frac{1}{\sqrt{2\pi}} \int_A \sin nt\,dt\right).$$

Then by the Riemann-Lebesgue lemma, μ is well defined and can be quickly shown to be countably additive and of finite variation. However, μ is not decomposable as it is easily seen to have nonrelatively compact range. It should be noted that $\mu(\Sigma)$ being nonrelatively compact also yields that μ is not Bochner differentiable with respect to Lebesgue measure [2].

After considering this example, one might suspect that the concepts of decomposability and differentiability of a measure with values in c_0 are closely related. Our final example, due to D. R. Lewis [7], indicates why this suspicion is ill-founded.

EXAMPLE. Let (Ω, Σ, μ) be any finite nonnegative measure space with μ nonatomic and consider each element x of c_0 as a doubly indexed sequence $x = (x_{n,i})$ where $n \geqslant 1$ and $2^n \leqslant i < 2^{n+1}$. Then the nonatomic nature of μ allows us to generate a sequence of measurable sets $(A_{n,i})$ such that $\mu(A_{n,i}) = 2^{-n}\mu(\Omega)$, where $A_{n,i}$ is the disjoint union of $A_{n+1,2i}$ and $A_{n+1,2i+1}$. If we now define $F\colon \Sigma \to c_0$ by

$$F(A) = (\mu(A \cap A_{n,i})),$$

then F is clearly μ-continuous and possesses finite variation. Furthermore, in [7] Lewis has shown that the range of F is relatively compact, convex, and that F has no Bochner derivative with respect to μ. Thus, we have a decomposable, nondifferentiable vector measure of finite variation with values in c_0.

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