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AN EXAMPLE OF A DOUBLY CONNECTED DOMAIN WHICH ADMITS A QUADRATURE IDENTITY

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ABSTRACT. In this paper we construct a doubly connected domain $D \ni 0$ such that $\iint_D f(z)\,d\sigma = Af(0) + Bf'(0)$ for any analytic and area integrable in D function f, which has a single-valued integral in D.

1. **Introduction.** We first introduce the notation (see [1]). Let D be a bounded plane domain. By $L_a^1(D)$ we denote the set of single-valued analytic functions in D which are integrable in D with respect to the areal measure $d\sigma$, and by $L_{a,s}^1(D)$ the subset of $L_a^1(D)$ consisting of functions with single-valued integral. We say that D admits a quadrature identity (q.i.) relative to $L_a^1(D)$ (or $L_{a,s}^1(D)$) if there exist a point $z_0 \in D$ and complex numbers A, B such that

$$(*) \qquad \int_D f\,d\sigma = Af'(z_0) + Bf(z_0)$$

for every $f \in L_a^1(D)$ (or $f \in L_{a,s}^1(D)$).

For a discussion of the background of this problem, see [1]. We note only that for a one point q.i., namely

$$(**) \qquad \int_D f\,d\sigma = Af(z_0)$$

there is no difference between L_a^1 and $L_{a,s}^1$. It can be shown [1, Theorem 7] that the validity of $(**)$ for every $f \in L_{a,s}^1(D)$ implies that D is simply connected and, hence, a disc centered at z_0.

In the present paper we show that the validity of $(*)$ for all f in $L_{a,s}^1(D)$ does not imply that D is a simply connected domain. We prove the following

THEOREM. *There exists a bounded doubly connected domain D which admits a quadrature identity* $(*)$ *for all* $f \in L_{a,s}^1(D)$.

REMARKS. 1. It turns out that the validity of $(*)$ for all $f \in L_a^1(D)$ does imply that D is simply connected. This fact was proved by D. Aharonov and H. Shapiro [1, Theorem 4]. Such a domain D can be found explicitly.

2. Our theorem is closely related to a certain minimal-area problem considered in [2]. In fact, our example shows that the method in [2], as it stands now, is not sufficiently strong to conclude that a certain domain D is

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simply connected. Such a conclusion would lead to the complete solution of the above-mentioned extremal problem.

3. A recent result (unpublished) of B. Gustafsson shows that if we consider a q.i. of higher order, then for every positive integer n there exists a domain of connectivity n which admits a q.i. of some (unknown) order.

2. We start with some preliminary results.

LEMMA 1. *Let D be a bounded doubly connected domain. Let its boundary be $C = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are nonintersecting rectifiable Jordan curves, Γ_2 surrounding Γ_1. Then the rational functions from $L^1_{a,s}(D)$ are dense in $L^1_{a,s}(D)$ in the $L^1(D)$-metric.*

PROOF. Let ζ be some fixed point in the interior of Γ_1. It is known[1] that under our topological requirement on ∂D it is possible to approximate (in the $L^1(D)$-metric) every function $f \in L^1_a(D)$ by rational functions with poles at ζ and at infinity. Let $R_n(z)$ be a sequence of such rational functions for a given f, i.e. $\int_D |f(z) - R_n(z)|\,d\sigma \to 0$ as $n \to \infty$. It is known (see [4, p. 109]) that this condition implies the uniform convergence of $R_n(z)$ to $f(z)$ on any closed set interior to D. Consequently, we have for some fixed path $\gamma \subset D$ which surrounds Γ_1:

$$\left|\int_\gamma R_n(z)\,dz\right| = \left|\int_\gamma (R_n(z) - f(z))\,dz\right| \leqslant c_\gamma \max_\gamma |f(z) - R_n(z)| \to 0.$$

This means that the residue α_n of $R_n(z)$ at point ζ tends to zero. Let $\tilde{R}_n(z) = R_n(z) - \alpha_n/(z - \zeta)$; then $\tilde{R}_n(z) \in L^1_{a,s}(D)$ and we have

$$\int_D |f - \tilde{R}_n|\,d\sigma \leqslant \int_D |f - R_n|\,d\sigma + |\alpha_n| \int_D \frac{d\sigma}{|z - \zeta|} \to 0.$$

LEMMA 2. *Let D be as in Lemma 1 and contain zero, and let $f(z)$ be continuous in $\overline{D}$ and analytic in D except for a double pole at $z = 0$, and such that*

$$f(z) = \begin{cases} \bar{z} & \text{on } \Gamma_1 \\ \bar{z} + \lambda & \text{on } \Gamma_2 \quad (\lambda \text{ being some complex number}). \end{cases}$$

Then D admits a q.i. $(*)$ for every $h \in L^1_{a,s}(D)$.

PROOF. Set $f(z) = a/z^2 + b/z + T(z)$ where $T(z)$ is analytic in D. Let $h(z)$ be continuous in $\overline{D}$ and $h(z) \in L^1_{a,s}(D)$. This means that $\int_{\Gamma_1} h\,dz = \int_{\Gamma_2} h\,dz = 0$. Hence, using Green's formula,

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[1] See, for instance, [3, p. 114], where this result is formulated (in much stronger form) for the $L^2(D)$-metric. The proof is based on the fact that any function in $L^2_a(G)$ (G is a simply connected bounded Jordan domain) can be approximated in the $L^2(D)$-metric by polynomials. But this last assertion is also true for the case of $L^1(D)$-approximation (see, for instance, [4, p. 45]). So the proof in [3] holds for our case as well.

$$\int_D h\,d\sigma = \frac{1}{2i}\int_C \bar{z}h(z)\,dz = \frac{1}{2i}\int_{\Gamma_1} f(z)h(z)\,dz + \frac{1}{2i}\int_{\Gamma_2}[f(z)-\lambda]h(z)\,dz$$
$$= \frac{1}{2i}\int_C f(z)h(z)\,dz = \frac{1}{2i}\int_C\left(\frac{a}{z^2}+\frac{b}{z}+T(z)\right)h(z)\,dz = Ah'(0)+Bh(0)$$

(where $A = \pi a$, $B = \pi b$). Thus, the q.i. holds for $h \in L^1_{a,s}(D)$ provided h is continuous in $\overline{D}$. Let now $h(z)$ be any function in $L^1_{a,s}(D)$. In view of Lemma 1, we can find a sequence $R_n(z)$ such that $\int_D h\,d\sigma = \lim_{n\to\infty}\int_D R_n(z)\,d\sigma$, each $R_n(z)$ having a single-valued integral in D.

Consequently we have

$$\int_D h(z)\,d\sigma = \lim_{n\to\infty}\int_D R_n(z)\,d\sigma = \lim_{n\to\infty}(AR'_n(0)+BR_n(0))$$
$$= Ah'(0)+Bh(0).$$

LEMMA 3. *Let Δ be a closed annulus $1 \leqslant |z| \leqslant R$ and let $1 < \alpha < R$. Then for every real λ there exist a function $g(z)$ which is analytic in Δ, and another function $f(z)$ which is analytic in Δ except for a double pole at $z = \alpha$, such that*

$$f(z) = \begin{cases} \overline{g}(z) & \text{on } |z| = 1, \\ \overline{g}(z)+\lambda & \text{on } |z| = R. \end{cases}$$

PROOF. Let $f(z) = a/(z-\alpha)^2 + b/(z-\alpha) + T(z)$ where a, b are real and $T(z)$ is analytic in Δ. We then have the expansions for f:

$$\begin{aligned} f(z) &= \frac{a}{\alpha^2}\sum_0^\infty (k+1)\left(\frac{z}{\alpha}\right)^k - \frac{b}{\alpha}\sum_0^\infty\left(\frac{z}{\alpha}\right)^k \\ &\quad + \sum_1^\infty T_k z^k + \sum_1^\infty T_{-k}z^{-k} + T_0 \quad \text{on } |z| = 1, \end{aligned} \tag{1}$$

$$\begin{aligned} f(z) &= \frac{a}{z^2}\sum_0^\infty (k+1)\left(\frac{\alpha}{z}\right)^k + \frac{b}{z}\sum_0^\infty\left(\frac{\alpha}{z}\right)^k \\ &\quad + \sum_1^\infty T_k z^k + \sum_1^\infty T_{-k}z^{-k} + T_0 \quad \text{on } |z| = R. \end{aligned} \tag{2}$$

For $g(z)$ which is analytic in Δ we have

$$\overline{g(z)} = \sum_1^\infty \bar{g}_k z^{-k} + \sum_1^\infty \bar{g}_{-k}z^k + \bar{g}_0 \quad \text{on } |z| = 1, \tag{3}$$

$$\overline{g(z)}+\lambda = \sum_1^\infty \bar{g}_k R^{2k}z^{-k} + \sum_1^\infty \bar{g}_{-k}R^{-2k}z^k + \bar{g}_0 + \lambda \quad \text{on } |z| = R. \tag{4}$$

Provided $f(z) = \overline{g(z)}$ on $|z| = 1$, we obtain

$$\frac{a}{\alpha^2} - \frac{b}{\alpha} + T_0 = \bar{g}_0;$$

(5)

$$(k+1)\frac{a}{\alpha^{k+2}} - \frac{b}{\alpha^{k+1}} + T_k = \bar{g}_{-k}, \quad T_{-k} = \bar{g}_k \qquad (k = 1,2,\dots).$$

Provided $f(z) = \overline{g(z)} + \lambda$ on $|z| = R$, we obtain

$$T_0 = \bar{g}_0 + \lambda, \quad (k-1)a\alpha^{k-2} + b\alpha^{k-1} + T_{-k} = \bar{g}_k R^{2k}, \quad T_k = R^{-2k}\bar{g}_{-k} \tag{6}$$

$$(k = 1,2,\dots).$$

From (5) and (6) we obtain

$$\lambda = b/\alpha - a/\alpha^2, \tag{7}$$

(8)

$$\bar{g}_k = g_k = \frac{a(k-1)\alpha^{k-2} + b\alpha^{k-1}}{R^{2k} - 1},$$

$$\bar{g}_{-k} = g_{-k} = \left(\frac{a}{\alpha^{k+2}}(k+1) - \frac{b}{\alpha^{k+1}}\right)\frac{R^{2k}}{R^{2k} - 1} \qquad (k = 1,2,\dots),$$

$$T_0 = \bar{g}_0 + \lambda,$$

$$T_k = \left(\frac{a}{\alpha^{k+2}}(k+1) - \frac{b}{\alpha^{k+1}}\right)\frac{1}{R^{2k} - 1}, \quad T_{-k} = \frac{a(k-1)\alpha^{k-2} + b\alpha^{k-1}}{R^{2k} - 1} \tag{9}$$

$$(k = 1,2,\dots).$$

Using (8), (9) we obtain after a simple manipulation:

(10)

$$f(z) = \frac{a}{(z-\alpha)^2} + \frac{b}{z-\alpha} + T_0 + \lambda \sum_{k=1}^{\infty} \frac{(\alpha/z)^k - (\alpha/z)^{-k}}{R^{2k} - 1}$$

$$+ \frac{a}{\alpha^2} \sum_{k=1}^{\infty} k \frac{(\alpha/z)^k + (\alpha/z)^{-k}}{R^{2k} - 1},$$

(11)

$$g(z) = \bar{T}_0 - \lambda + \lambda \sum_{k=1}^{\infty} \frac{(\alpha z)^k - R^{2k}(\alpha z)^{-k}}{R^{2k} - 1}$$

$$+ \frac{a}{\alpha^2} \sum_{k=1}^{\infty} k \frac{(\alpha z)^k + R^{2k}(\alpha z)^{-k}}{R^{2k} - 1}.$$

We note, that for the present T_0 may be chosen arbitrarily; a and b may also be arbitrary (and real) but must satisfy (7). Since $1 < \alpha < R$, $g(z)$ and the regular part of $f(z)$ are analytic in the closed annulus Δ and the above computation (5), (6) shows, that f and g satisfy the requirement of Lemma 3.

LEMMA 4. *The function* $g(z)$ *which is defined by* (11) *is univalent in the closed annulus* Δ, *provided* $\alpha = R^{3/4}$, $a = \lambda\alpha^2$ $(\lambda \neq 0)$, *and provided* R *is sufficiently large.*

PROOF. Provided $\lambda = a/\alpha^2$, we obtain from (11): $g(z) = \overline{T}_0 - \lambda + \lambda\phi(z)$, where we set

$$\phi(z) = \sum_1^\infty \frac{(\alpha z)^k - R^{2k}(\alpha z)^{-k}}{R^{2k}-1} + \sum_1^\infty k\frac{(\alpha z)^k + R^{2k}(\alpha z)^{-k}}{R^{2k}-1}. \tag{12}$$

Provided $\lambda \neq 0$, it suffices to prove univalence of $\phi(z)$. We obtain from (12):

$$\begin{aligned}\phi(z) &= \frac{2\alpha z}{R^2-1} + \sum_2^\infty \frac{(\alpha z)^k - R^{2k}(\alpha z)^{-k}}{R^{2k}-1} + \sum_2^\infty k\frac{(\alpha z)^k + R^{2k}(\alpha z)^{-k}}{R^{2k}-1}\\ &= \frac{2\alpha z}{R^2-1} + \phi_1(z).\end{aligned} \tag{13}$$

For any pair of points z_1, z_2 in Δ we have

$$\phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} \phi'(z)\,dz = \frac{2\alpha}{R^2-1}(z_2 - z_1) + \int_{z_1}^{z_2} \phi_1'(z)\,dz.$$

We can choose the path of integration from z_1 to z_2 in such a way that its length will not exceed $3|z_1 - z_2|$. Thus, in order to prove univalence of $\phi(z)$ we have to show that

$$\max_{z\in\Delta} |\phi_1'(z)| < 2\alpha/3(R^2-1). \tag{14}$$

From (13) we obtain

$$\begin{aligned}|\phi_1'(z)| &\leqslant \alpha\left\{\sum_{k=2}^\infty k\frac{|\alpha z|^{k-1} + R^{2k}|\alpha z|^{-k-1}}{R^{2k}-1} + \sum_{k=2}^\infty k^2\frac{|\alpha z|^{k-1} + R^{2k}|\alpha z|^{-k-1}}{R^{2k}-1}\right\}\\ &< 2\alpha\sum_{k=2}^\infty k^2\frac{|\alpha z|^{k-1} + R^{2k}|\alpha z|^{-k-1}}{R^{2k}-1}\\ &< 4\alpha\sum_{k=2}^\infty k^2\frac{|\alpha z|^{k-1} + R^{2k}|\alpha z|^{-k-1}}{R^{2k}}[2]\\ &= \frac{4\alpha}{R^2}\sum_2^\infty k^2\left|\frac{\alpha z}{R^2}\right|^{k-1} + \frac{4\alpha}{|\alpha z|^2}\sum_2^\infty k^2\frac{1}{|\alpha z|^{k-1}}.\end{aligned} \tag{15}$$

If we choose now $\alpha = R^{3/4}$, we obtain from (15):

$$|\phi_1'(z)| \leqslant 4R^{-5/4}\sum_2^\infty k^2(R^{-1/4})^{k-1} + 4R^{-3/4}\sum_2^\infty k^2(R^{-3/4})^{k-1} < CR^{-3/2}$$

for sufficiently large R.

The validity of (14) for $\alpha = R^{3/4}$ is now clear, which proves the lemma.

[2] This step is correct provided $R > 2^{1/4}$.

3. We now proceed to prove our theorem. We choose some real λ ($\lambda \neq 0$) and some sufficiently large R (for which Lemma 4 is valid). Set $\alpha = R^{3/4}$, $a = \alpha^2 \cdot \lambda$, $b = 2\lambda\alpha$. The function $g(z)$ is then defined by (11) up to the additive constant $\overline{T}_0$, chosen such that $g(R^{3/4}) = 0$. In view of Lemma 4, $w = g(z)$ is analytic and univalent in the closed annulus $\Delta = \{1 \leqslant |z| \leqslant R\}$ and maps its interior onto the doubly connected domain D, which contains zero. Set $\tilde{f}(w) = f(g^{-1}(w))$, where f is defined by (10) (with λ, R, α, a, b, T_0 as chosen above). $\tilde{f}(w)$ is single valued and analytic in D except for a double pole at $w = 0$. By virtue of Lemma 3, we have

$$\tilde{f}(w) = \begin{cases} \overline{w} & \text{on } \Gamma_1 = \{w\colon w = g(z), |z| = 1\}, \\ \overline{w} + \lambda & \text{on } \Gamma_2 = \{w\colon w = g(z), |z| = R\}. \end{cases}$$

Since $g(z)$ is analytic and univalent in the closed annulus, the curves Γ_1 and Γ_2 satisfy the topological conditions of Lemma 1 and, hence, by Lemma 2, the domain D admits a quadratus identity (*) with $z_0 = 0$.

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