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VANISHING SOLUTIONS OF THE DISSIPATIVE ACOUSTIC EQUATION IN AN EXTERIOR DOMAIN

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ABSTRACT. Except in one dimension, strictly incoming waves cannot be propagated by the wave equation with dissipative boundary conditions so that they disappear asymptotically in forward time.

In [4] Lax and Phillips consider the acoustic equation in an exterior domain $G \subset \mathbf{R}^n$:

$$\begin{cases} u_{tt} = \Delta u & \text{in } G, \\ \partial_n u + \sigma u_t = 0, & \sigma \geqslant 0 \text{ in } \partial G. \end{cases} \tag{1.1}$$

They assume G contains the complement of the ball of radius ρ. As in [4], we define H to be the Hilbert space of all initial data d with finite energy in G. Let $T(t)$ be the (strongly continuous) contraction semigroup formed by mapping initial data into data at time t.

If $G = \mathbf{R}^n$ (and the second part of (1.1) is vacuous) we will denote H by H_0 and $T(t)$ by $U_0(t)$. We note that $U_0(t)$ is a unitary group. We denote the cogenerator (see Chapter 3 of [5]) of $T(t)$ by T and the cogenerator of $U_0(t)$ by U_0. Let $D_\pm \subset H$ be the set of all initial data vanishing on $\{x \mid |x| \leqslant \rho \pm t, t \geqslant 0\}$.

We will prove the following

THEOREM. *Let n be greater than* 1. (*Recall that $G \subset \mathbf{R}^n$.*) *If $d \in D_-$ and $d \not\equiv 0$. Then* $\lim_{t \to +\infty} T(t)d \neq 0$.

Before starting the proof we recall some of the material in [2], [3], and [4]. We represent the action of $U_0(t)$ on H_0 as right translation on $L^2(\mathbf{R}, N)$ (i.e., the set of all square integrable N-valued functions on $\mathbf{R}$) for some auxiliary Hilbert space N so that D_- is mapped onto $L^2(\mathbf{R}_- - \rho, N)$. In this representation D_+ is mapped onto

$$L^2(\mathbf{R}_+ + \rho, N) \text{ if } n \text{ is odd}$$

and

$$\mathcal{H} L^2(\mathbf{R}_+ + \rho, N) \text{ if } n \text{ is even}$$

where

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$$\mathcal{H}(s) = \mathfrak{F}^{-1}\mathcal{H}(\sigma)\mathfrak{F}, \tag{1.2}$$

$$\mathcal{H}(\sigma) = \operatorname{sgn} \sigma \tag{1.3}$$

and $\mathfrak{F}$ is the Fourier transform.

Since $T(t)|_{D_+} = U_0(t)|_{D_+}$ for $t \geqslant 0$, and $T(t)^*|_{D_-} = U_0(-t)|_{D_-}$ for $t \geqslant 0$, we can embed H onto $L^2(\mathbf{R}, N)$ so that $T(t)^*$ acts on $L^2(\mathbf{R}_- - \rho, N)$ as left translation by t and $T(t)$ acts on $L^2(\mathbf{R}_+ + \rho, N)$ (resp. $\mathcal{H}L^2(\mathbf{R}_+ + \rho, N)$) if $n =$ odd (resp. if $n =$ even) as right translation by t. The action of $T(t)$ on the rest of $L^2(\mathbf{R}, N)$ is more difficult to describe.

LEMMA 1.1. *Let $f(s) \in D_-$. Then $f(s) \in T^*D_-$ if and only if $\hat{f}(\sigma)$, the Fourier transform of $f(s)$, is zero at the point* $(0, -i)$.

PROOF. Let $f(s) \in D_-$. Then by Chapter III of [5] and the fact that $T(t)^*f(s) = f(s + t)$ for $t \in \mathbf{R}_+$ we conclude

$$(T^*f)(s) = f(s) \ \operatorname*{s-lim}_{t\to 0^+} \frac{t}{1+t} \sum_{n=0}^{\infty} \frac{f(s+nt)}{(1+t)^n}.$$

Taking the Fourier transform

$$\begin{aligned}\widehat{(T^*f)}(\sigma) &= \hat{f}(\sigma) + \operatorname*{s-lim}_{t\to 0^+} \frac{t}{1+t} \sum_{n=0}^{\infty} \frac{e^{int\sigma}\hat{f}(\sigma)}{(1+t)^n} \\ &= \hat{f}(\sigma)\left(1 + \operatorname*{s-lim}_{t\to 0^+} \frac{t}{1+t} \sum_{n=0}^{\infty} \frac{e^{int\sigma}}{(1+t)^n}\right) \\ &= \hat{f}(\sigma)(1 - 1/i\sigma).\end{aligned}$$

Since $\widehat{(T^*f)}(\sigma)$ and $\hat{f}(\sigma)$ are analytic in the lower half plane, the above calculation shows $(Tf)(-i) = 0$.

Conversely if $g(s) \in D_-$ and $\hat{g}(\sigma)$ has a zero at $-i$, then

$$\hat{g}(\sigma) = (\sigma + i)(\sigma - i)^{-1}\hat{f}(\sigma) \quad \text{for some } f \in D_-.$$

But $T^* = U_0^{-1}$ on D_-, and U_0^{-1} acts as multiplication by $(\sigma + i)/(\sigma - i)^{-1}$ in the Fourier transform of the translation representation (called the spectral representation in [2]). To see this, note that A_0, the generator of $U_0(t)$, acts as multiplication by $i\sigma$ in the spectral representation. The action of

$$U_0 = (I + A_0)(I - A_0)^{-1}$$

is now clear. Thus $g(s) = (T^*f)(s)$ for $f \in D_-$. This proves the lemma.

Define the wave operators as

$$W_1 = \operatorname*{s-lim}_{t\to\infty} T(t)J_0U_0(-t), \qquad W_2 = \operatorname*{s-lim}_{t\to\infty} U_0(-t)JT(t), \tag{1.4}$$

where J, J_0 are continuous linear maps from H to H_0 and H_0 to H respectively which act as the identity on $D_- \vee D_+$. Define the scattering operator S as in [4] by

$$S = W_2W_1. \tag{1.5}$$

LEMMA 1.2. *For any $d \in D_-$*

$$P_{D_+}Td = P_{D_+}U_0Sd. \tag{1.6}$$

PROOF. From the definitions of W_1 and W_2 we have $W_2T = U_0W_1$. Since $W_2|_{D_+} = I|_{D_+} = W_2^*|_{D_+}$ we see that for any $d \in H$,

$$P_{D_+}U_0W_2d = P_{D_+}W_2Td = W_2P_{D_+}Td = P_{D_+}Td.$$

If $d \in D_-$ we see that $W_1d = d$ so that by (1.5)

$$P_{D_+}Td = P_{D_+}U_0[W_2W_1]d = P_{D_+}U_0Sd$$

for any $d \in D_-$. Q.E.D.

Since $U_0(t)$ acts as right translation by t on $L^2(\mathbf{R}, N)$ we can calculate U_0 as

$$(U_0f)(s) = f(s) - 2e^{-s}\int_{-\infty}^{s} f(\zeta)e\zeta\, d\zeta, \qquad f \in L^2(\mathbf{R}, N). \tag{1.7}$$

The operator S on $H_0 = L^2(\mathbf{R}, N)$ commutes with $U_0(t)$ (= translation by t) and it follows that in the spectral representation (= Fourier transform space) the corresponding operator, denoted by $\mathcal{S}$, acts on $L^2(\mathbf{R}, N)$ by multiplication

$$(\mathcal{S}f)(\sigma) = \mathcal{S}(\sigma)f(\sigma), \qquad f \in L^2(\mathbf{R}, N).$$

We now prove the theorem for the case when n is odd ($\neq 1$). In [4] it is shown that $\mathcal{S}(\sigma)$ has an analytic extension to the lower half plane if n is odd. In particular it is shown that

$$S\big(L^2(\mathbf{R}_- - \rho, N)\big) \subset L^2(\mathbf{R}_- + \rho, N). \tag{1.8}$$

PROPOSITION 1.3. *Let d be a nonzero element of D_-. We also assume $U_0d \not\in D_-$ and*

$$\mathcal{S}(-i) \text{ is invertible.} \tag{1.9}$$

Then U_0Sd is not orthogonal to D_+.

PROOF. Let $d \in D_-$. Then in the translation representation d has its support in $(-\infty, -\rho]$. Since S satisfies (1.8) we see (Sf) has its support in $(-\infty, \rho]$. From (1.7) it is clear that if U_0Dd has its support in $(-\infty, \rho]$ then

$$0 = \int_{-\infty}^{\rho} (Sd)(\zeta)e^{\zeta}\, d\zeta = \int_{-\infty}^{\infty} (Sd)(\zeta)e^{\zeta}\, d\zeta. \tag{1.10}$$

Rewriting (1.10) we see $\widehat{(Sd)}(-i) = 0$, i.e. $\mathcal{S}(-i)\hat{d}(-i) = 0$. By assumption, $\mathcal{S}(-i)$ is invertible and we conclude $\hat{d}(-i) = 0$. Thus by Lemma 1.1 we see $d \in T^*D_- = U_0^{-1}D_-$, i.e. $U_0d \in D_-$. But we assumed $U_0d \not\in D_-$. Thus U_0Sd does not have its support in $(-\infty, \rho]$.

Since $D_+ = L^2((\rho, \infty), N)$ in the translation representation, we conclude that U_0Sd is not orthogonal to D_+.

PROPOSITION 1.4. *Let $d \in D_-$ be nonzero and assume* (1.9) *holds. Then* $\text{s-lim}_{t\to\infty} T(t)d \neq 0$.

PROOF. By Proposition III 9.1 of [5], it suffices to show

(1.11) $$\operatorname*{s-lim}_{n\to\infty} T^n d \neq 0 \quad \text{for all } d \in D_-.$$

Now if $d \not\equiv 0$ we can find a smallest $m \geqslant 0$ so that $T^m d \not\in T^* D_- = U_0^{-1} D_-$, and $T^m d \in D_-$. We conclude by Proposition 1.3 that $U_0 S T^m d$ is not orthogonal to D_+. Thus by (1.6) we see $P_{D_+} T^{m+1} d \neq 0$. Now let U on $K \supset H$ be the minimal unitary dilation of T (see [5]). On D_+ we see $T|_{D_+} = U_0|_{D_+} = U|_{D_+}$. Thus for $n \geqslant 0$

$$\begin{aligned} 0 &= (D_+, H \ominus D_+) = (U^n D_+, U^n (H \ominus D_+)) \\ &= (T^n D_+, U^n (H \ominus D_+)) = (T^n D_+, T^n (H \ominus D_+)). \end{aligned}$$

Thus if $T^{m+1} d = \beta \oplus \beta_+$, $\beta \in H \ominus D_+$, $\beta_+ \in D_+$ we see

$$T^n \beta_+ \perp T^n \beta \quad \text{all } n \geqslant 0.$$

Thus

$$\begin{aligned} \|T^n T^{m+1} d\|^2 &= \|T^n \beta_+ + T^n \beta\|^2 = \|T^n \beta_+^2 + T^n \beta\|^2 \geqslant \|T^n \beta_+\|^2 \\ &= \|U_0^n \beta_+\|^2 = \|\beta_+\|^2 = \|P_{D_+} T^{m+1} d\|^2 > 0. \end{aligned}$$

Thus we can conclude (1.11). Q.E.D.

We now relax the restriction imposed by (1.9) and complete the proof of the theorem in the odd-dimensional case.

PROPOSITION 1.5. *If $d \in D_-$, then*

(1.12) $$\operatorname*{s-lim}_{t\to\infty} T(t) d \neq 0.$$

PROOF. Recall that G contains the complement of the ball of radius ρ. Define $V(x, t) = u(cx, ct)$, $c > 0$. Then $u(x, t)$ satisfies

(1.13) $$\begin{cases} v_{tt} = \Delta v & \text{in } G', \\ \partial_n v + \sigma v_t = 0 & \text{in } \partial G', \sigma \geqslant 0, \end{cases}$$

where $G' =^{\text{def}} \{c^{-1} g \mid g \in G\}$.

Define $D_-(v)$ as the subspace of initial data which vanishes on $\{|x| \leqslant \rho/c + t, t \leqslant 0\}$ under the action of (1.13). Recall the definition of $D_-(u)$ as the subspace of initial data which vanishes on $\{|x| \leqslant \rho + t, t \leqslant 0\}$ under the action of (1.1). It is clear that c can be chosen so that G' contains the complement of a ball of radius less than one. By Theorem 10.10 of [4], since n is greater than one, we can conclude that the scattering matrix for the v-system is invertible at $-i$. Thus by Propositions 1.3 and 1.4, (1.12) holds for the v-system. But the statement of the theorem is invariant under the change from the v to the u systems. Thus (1.12) holds for both the u and v systems and the theorem is proven for the case when n is odd and greater than one.

We now look at the case when n is even. To prove the theorem in this case it suffices to establish that $U_0 S d$ is not orthogonal to D_+ for any nonzero d in D_-. Once this is done the argument in Proposition 1.4 (with $m = 0$) can be used as before to conclude (1.11).

PROPOSITION 1.6. *Let d be a nonzero element of D_- and let n be even. Then U_0Sd is not orthogonal to D_+.*

PROOF. Let $d \in D_-$. Then

$$(Sd, D_+) = (W_2W_1d, D_+) = (W_1d, W_2^*D_+) = (d, D_+).$$

If Sd is orthogonal to D_+, then $d \in D_- \cap D_+\perp$. Thus $\hat{d}(\sigma)$ and $\mathcal{K}(\sigma) \cdot \hat{d}(\sigma)$ both have analytic extensions to the lower half plane. But this is clearly impossible unless $\hat{d}(\sigma) \equiv 0$, i.e. $d(s) \equiv 0$. Thus Sd is not orthogonal to D_+. Since $U_0^{-1}D_+ \supset D_+$ we conclude U_0Sd is not orthogonal to D_+.

The proof of the theorem is now complete.

In conclusion, I would like to thank the referee for pointing out that the theorem does not hold for $n = 1$, by providing the following counterexample:

$$G = \{x > a\}, \qquad u = f(x + t), \qquad f \text{ of compact support,}$$
$$-u_x + u_t = 0 \quad \text{on } x = a.$$

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