PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

# INDUCTION ON SYMMETRIC AXIAL MAPS AND EMBEDDINGS OF PROJECTIVE SPACES

A. J. BERRICK

ABSTRACT. A homotopy class of axial maps $P^n \times P^n \to P^{n+k}$ determines an invariant in $\pi_n(V_{n+k+1,n+1})$ $(2k \geqslant n+2)$. If an axial map is symmetric and has trivial invariant it extends to a symmetric axial map $P^{n+1} \times P^{n+1} \to P^{n+k+1}$. An immersion of $P^n$ in $R^{n+k}$ lifts to an immersion of $S^n$ in $R^{n+k}$ and so has a Smale invariant. For $j$: $R^{n+k} \hookrightarrow R^{n+k+2}$, $2k \geqslant n+2$ (resp. $2k \geqslant n+3$), any embedding $a$: $P^n \to R^{n+k}$ with trivial Smale invariant induces an embedding of $P^{n+1}$ in $R^{n+k+2}$ whose restriction to $P^n$ is regularly homotopic (resp. isotopic) to $ja$.

Let $P^n$ denote real projective $n$-space. A map $P^n \times P^n \to P^{n+k}$ is *axial* if it is homotopic to the inclusion $P^n \to P^{n+k}$ when composed with either inclusion $P^n \to * \times P^n \to P^n \times P^n$, $P^n \to P^n \times * \to P^n \times P^n$, and *symmetric axial* if also it is equivariant with respect to the interchange involution on $P^n \times P^n$ and the trivial involution on $P^{n+k}$. This note defines an invariant of the homotopy class of such a map and examines the consequences of its vanishing.

Let $f$: $S^n \times S^n \to S^{n+k}$ have $f(x,-y) = -f(x,y)$. Then exponential correspondence yields a map $f'$: $S^n \to Y_{n+k+1,n+1}$, where $Y_{r,s}$ is the space of antipodal-involution-preserving maps from $S^{s-1}$ to $S^{r-1}$ introduced in [5, §1]. For $2k \geqslant n+2, f'$ lifts by [5, (1.1)] to a map into the Stiefel manifold $V_{n+k+1,n+1}$, whose homotopy class $\phi_f \in \pi_n(V_{n+k+1,n+1})$ is an invariant of the equivariant homotopy class of $f$. If $f$ covers an axial map $\tilde{f}$: $P^n \times P^n \to P^{n+k}$, we write $\phi_{\tilde{f}}$ equally for $\phi_f$.

LEMMA 1. *Suppose $2k \geqslant n+2$. Let $\tilde{f}$ be a symmetric axial map $P^n \times P^n \to P^{n+k}$ whose invariant $\phi_{\tilde{f}} \in \pi_n(V_{n+k+1,n+1})$ is the trivial element. Then $i \circ \tilde{f}$ extends to a symmetric axial map $\tilde{g}$: $P^{n+1} \times P^{n+1} \to P^{n+k+1}$ ($i$: $P^{n+k} \hookrightarrow P^{n+k+1}$). Further, if symmetric axial $\tilde{f}'$: $P^n \times P^n \to P^{n+k}$ is homotopic to $\tilde{f}$, then $i \circ \tilde{f}'$ extends to $\tilde{g}'$ homotopic to $\tilde{g}$; if $\tilde{f}'$ is moreover symmetric homotopic to $\tilde{f}$ (i.e. via a symmetry-preserving homotopy), then $\tilde{g}'$ is symmetric homotopic to $\tilde{g}$.*

PROOF. Let $f$: $S^n \times S^n \to S^{n+k}$ cover $\tilde{f}$. The condition ensures that $f$ may be extended to $f_1$: $CS^n \times S^n \to S^{n+k}$, where $CS^n = (S^n \times I)/(S^n \times \{1\})$. Make the identification $S^{N+1} = CS^N \cup (S^N \times [0,-1])/(S^N \times \{-1\})$. Then $f_1$ ex-

Received by the editors October 13, 1975.
*AMS (MOS) subject classifications* (1970). Primary 55E05, 57D40.
*Key words and phrases.* Axial map, embedding, immersion, isotopy, projective space, regular homotopy, Smale invariant, Stiefel manifold.

tends to $f_2$: $CS^n \times CS^n \to S^{n+k+1}$ by

$$f_2([x,s],[y,t]) = \begin{cases} [f_1([x,(s-t)/(1-t)],y),t], & t \leqslant s, \\ [f_1([y,(t-s)/(1-s)],x),s], & s \leqslant t. \end{cases}$$

$f_2$ extends to $g$: $S^{n+1} \times S^{n+1} \to S^{n+k+1}$ by means of

$$g([u,-s],[v,t]) = -g([-u,s],[v,t]), \quad s \geqslant 0,$$
$$g([u,s],[v,-t]) = -g([u,s],[-v,t]), \quad t \geqslant 0.$$

Clearly $g$ gives rise to a symmetric axial map $\tilde{g}$: $P^{n+1} \times P^{n+1} \to P^{n+k+1}$. The required homotopy properties are easily verified.

*Note.* This result generalises as follows (using the terminology of [2]). *If a homotopy class of symmaxial maps of type* $(n, k)$ *has trivial invariant in* $\pi_n(V_{n+k+1,n+1})$, *then it extends to a homotopy class of symmaxial maps of type* $(n + 1, k)$. Needless to say, the details of the proof are rather technical.

Let $a$: $S^n \to R^{n+k}$ be an immersion. Its differential map of tangent bundles induces (upon trivialising the stable tangent bundle of $S^n$, as in [1, §3]) a map $f_a$: $S^n \times S^n \to S^{n+k}$ which is *linear* on the second factor, thus yielding an invariant $\phi_a \in \pi_n(V_{n+k+1,n+1})$ of the regular homotopy class of $a$. $\phi_a$ is defined for arbitrary $n$, $k$. However, when $2k \geqslant n + 2$, it follows from [5] that $\phi_a = \phi_{f_a}$, justifying our notation. This invariant is applied in [4, (3.8)], to deduce nonimmersion results for $P^n$ in $R^{n+k}$. In the particular case where $a$ is the standard embedding $s$: $S^n \hookrightarrow R^{n+1} \hookrightarrow R^{n+k}$, it is readily seen that $\phi_s = 0$.

For immersions $a$ of $S^n$ in $R^{n+k}$ there is another, better known, regular homotopy invariant, viz. the *Smale invariant* $\Omega(a,s) \in \pi_n(V_{n+k,n})$ of [7]. Let $j$: $V_{n+k,n} \hookrightarrow V_{n+k+1,n+1}$ be the inclusion map. Then $j_* \Omega(a,s) = \phi_a - \phi_s$ by [6, (3.1)]. However $j_*$ is an isomorphism for $k > 1$.

LEMMA 2. *If* $a$: $S^n \to R^{n+k}$ *is an immersion* $(k > 1)$, *then its Smale invariant vanishes precisely when* $\phi_a$ *vanishes.*

Now any immersion of $P^n$ in $R^{n+k}$ lifts to an immersion of $S^n$ in $R^{n+k}$, so we may define its Smale invariant to be that of the lifting. Combining the above lemmas with [3] establishes the following result.

THEOREM 3. *For* $2k \geqslant n + 2$, *any regular homotopy class of embeddings of* $P^n$ *in* $R^{n+k}$ *with trivial Smale invariant extends to a regular homotopy class of embeddings of* $P^{n+1}$ *in* $R^{n+k+2}$; *for* $2k \geqslant n + 3$, *any isotopy class of embeddings of* $P^n$ *in* $R^{n+k}$ *with trivial Smale invariant extends to an isotopy class of embeddings of* $P^{n+1}$ *in* $R^{n+k+2}$.

Alternatively, we may use [1, (1.4)] instead of Theorem (1.2) of [3]. Then the following result may be deduced from the combination either of Lemmas 1, 2 above with [3, (1.1)] and [1, (1.4)], or of Lemma 2 with the generalisation of Lemma 1 cited above, [2, Theorem 1] and [1, (1.4)].

THEOREM 4. *If there exists an embedding of $P^n$ in $R^{n+k}$ with trivial Smale invariant, then $P^{n+1}$ immerses in $R^{n+k+1}$.*

It is amusing to note by contrast that for $2k \geqslant \max(4, n+2)$ any embedding of $S^n$ in $R^{n+k}$ extends to an immersion of $S^{n+1}$ in $R^{n+k}$. This follows from [6] and [7, Theorem E].

## REFERENCES

**1.** J. Adem, S. Gitler and I. M. James, *On axial maps of a certain type*, Bol. Soc. Mat. Mexicana (2) **17** (1972), 59–62. MR **49** #1530.

**2.** A. J. Berrick, *Axial maps with further structure*, Proc. Amer. Math. Soc. **54** (1976), 413–416.

**3.** A. J. Berrick, S. Feder and S. Gitler, *Symmetric axial maps and embeddings of projective spaces*, Bol. Soc. Mat. Mexicana (to appear).

**4.** S. Gitler, *The projective Stiefel manifolds*. II. *Applications*, Topology **7** (1968), 47–53. MR **36** #3373b.

**5.** A. Haefliger and M. W. Hirsch, *Immersions in the stable range*, Ann. of Math. (2) **75** (1962), 231–241. MR **26** #784.

**6.** M. A. Kervaire, *Sur l'invariant de Smale d'un plongement*, Comment. Math. Helv. **34** (1960), 127–139. MR **22** #4068.

**7.** Stephen Smale, *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. (2) **69** (1959), 327–344. MR **21** #3862.

ST. JOHN'S COLLEGE (OXFORD UNIVERSITY), OXFORD OX1 3JP, ENGLAND