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A ONE-SIDED SUMMATORY FUNCTION

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ABSTRACT. A method is given for summing one-sided series by employing the psi function. $\sum_{n=1}^{\infty} n^{-k}\Psi(n)$ is evaluated in closed form when $k \geqslant 2$ is an integer.

The use of the function $\pi \cot \pi z$ for the purpose of evaluating series of the form $\sum_{-\infty}^{\infty} f(n)$ is classical (cf. [2]). Most one-sided series however, even such simple ones as $\sum_{n=0}^{\infty} (n+1)^{-3}$, are beyond its reach. A function which does sum one-sided series is $\Psi(-z)$ where $\Psi(z) = \Gamma(z)/\Gamma'(z)$ and $\Gamma(z)$ denotes the gamma function. This is one consequence of Theorem 1, but probably not its most important one. More important, we believe, are the identities one obtains by a careful selection of admissible functions f. Some examples are given in Theorem 3.

Let $\mathcal{C}_n = \{z \in C \mid |z| = (n + \frac{1}{2})\}$ and let f be any meromorphic function none of whose poles lie on $\mathcal{C}_n$. (If any poles of f are on $\mathcal{C}_n$ we may alter $\mathcal{C}_n$ slightly so that the offending poles are inside the modified contour.) Then applying Cauchy's residue theorem we obtain

THEOREM 1. *If* $\lim_{n\to\infty}\{(2\pi i)^{-1}\int_{\mathcal{C}_n} f(z)\Psi(-z)\,dz\} = A$, *where* $|A| < \infty$, *then*

$$A = \sum \operatorname{Res}\,(f(z)\Psi(-z))$$

where the sum is taken over all the poles z_α *of* $f(z)\Psi(-z)$ *in the complex plane and the sum is ordered by* $|z_\alpha|$.

We now separate the poles of $f(z)\Psi(-z)$ into two disjoint sets S_1 and S_2 according to whatever result we wish to establish. Then we get

$$\sum_{S_1} \operatorname{Res}\,(f(z)\Psi(-z)) = A - \sum_{S_2} \operatorname{Res}\,(f(z)\Psi(-z))$$

where the sums may be given by the appropriate limits if necessary.

In the particular case where $f = p/q$ is a rational function with $\deg q \geqslant 2 + \deg p$ and none of the poles of f occur at the nonnegative integers the following well-known result obtains (cf. [1]).

THEOREM 2. *If the partial fraction decomposition of f is given by*

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$$\frac{p(z)}{q(z)} = \sum_i \frac{C_{i1}}{(z + a_i)} + \sum_i \frac{C_{i2}}{(z + a_i)^2} + \cdots + \sum_i \frac{C_{ik}}{(z + a_i)^k}$$

where $k = \{\max s|(z + a_i)^s$ *divides* $q(z)\} \leqslant \deg q$ *and the summations are over the zeros of* $q(z)$, *none of which occur at* 0 *or a negative integer, then*

$$\sum_{n=0}^{\infty} f(n) = -\sum_i C_{i1}\Psi(a_i) + \frac{1}{1!}\sum_i C_{i2}\Psi'(a_i) + \cdots + (-1)^k \frac{1}{(k-1)!}\sum_i C_{ik}\Psi^{(k-1)}(a_i). \tag{1}$$

Proof. The terms on the left-hand side of (1) come from the poles of $\Psi(-z)$ while those on the right arise from the poles of f. Thus it suffices to show that $\lim_{n\to\infty}\int_{\mathcal{C}_n} f(z)\Psi(-z)\,dz = 0$. Choose n sufficiently large to ensure that the zeros of $q(z)$ are inside $\mathcal{C}_n$ and then deform $\mathcal{C}_n$ into the square with vertices $(n + \frac{1}{2})(\pm 1 \pm i)$. Let A_n, B_n, C_n, D_n denote the vertices of the square beginning with A_n in the bottom left-hand corner and proceeding counterclockwise. In view of our assumptions we have $p(z)/q(z) = kz^{-2} + O(z^{-3})$ on the square. Since the asymptotic expansion

$$\Psi(z) \sim \log z - \frac{1}{2z} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2r} z^{-2r},$$

where the Bernoulli number B_{2r} is defined by

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} t^{2r},$$

is valid for $|\arg z| < \pi$, we estimate the integral on the three segments A_nB_n, C_nD_n, D_nA_n to be $O(n^{-1+\varepsilon})$ where $0 < \varepsilon < 1$. We obtain the same estimate on B_nC_n by employing the identity

$$\Psi(z) = \Psi(-z) - 1/z - \pi\cot\pi z.$$

Note that $\cot \pi z$ is bounded on B_nC_n. Q.E.D.

Remark 1. The purpose for giving the above proof is a twofold one. First of all, the only proofs of Theorem 2 that we know of employ finite difference techniques (cf. [5]), although D. H. Lehmer [3] recently obtained a restricted version by elementary means. Our main intention, however, was to indicate how one obtains estimates for $\int_{\mathcal{C}_n} f(z)\Psi(-z)\,dz$.

Remark 2. The following result is well known; $\zeta(3) = \sum_{n=0}^{\infty}(n+1)^{-3} = -\frac{1}{2}\Psi''(1)$, where ζ denotes the Riemann zeta function. It is a trivial consequence of Theorem 2.

It is clear that if f is a well-behaved meromorphic function, then $\sum_{-\infty}^{\infty} f(n) = \sum_{n=0}^{\infty}\{f(n) + f(-1-n)\}$. We are thus led to consider

$$\frac{1}{2\pi i}\int_{\mathcal{C}_n}\{f(z)+f(-1-z)\}\Psi(-z)\,dz = \frac{1}{2\pi i}\int_{\mathcal{C}_n} f(z)\{\Psi(-z)-\Psi(1+z)\}\,dz$$
$$= \frac{1}{2i}\int_{\mathcal{C}_n} f(z)\cot\pi z\,dz$$

as usual. Similarly many one- and two-sided series of the form $\sum g(n)f(n)$ where g is periodic, can be dealt with by the above methods. We note the following consequence of Theorem 2.

COROLLARY. *Let $g(n)$, $n \geqslant 0$, be any (real- or complex-valued) function which is periodic with period k and satisfies $\sum_{n=0}^{k-1} g(n) = 0$. Suppose that $f(z) = (z+a)^{-1}$ where $a \in \mathbf{C}$ and $a \neq 0$ or a negative integer. Then*

$$\sum_{n=0}^{\infty} f(n)g(n) = -\sum_{r=0}^{k-1}\frac{g(r)}{k}\Psi\left(\frac{a+r}{k}\right).$$

PROOF. We have

$$\sum_{r=0}^{k-1} g(r)\left(z+\frac{a+r}{k}\right)^{-1} = k\sum_{r=0}^{k-1} g(r)f(kz+r) = k\frac{p(z)}{q(z)}$$

where $q(z) = \prod_{r=0}^{k-1}(kz+a+r)$ and the polynomial

$$p(z) = \left(\sum_{r=0}^{k-1} g(r)\right)(kz)^{k-1} + \cdots$$

clearly has degree at most $k-2$. Since

$$\sum_{n=0}^{\infty} f(n)g(n) = \sum_{t=0}^{\infty}\sum_{r=0}^{k-1} g(r)f(kt+r),$$

the result follows easily from Theorem 2. This generalizes Theorem 8 of [3].

Let $f_k(z) = z^{-k}\Psi(-z)$. By much the same argument as above one can show that for each integer $k \geqslant 2$, $\lim_{n\to\infty}\int_{\mathcal{C}_n} f_k(z)\Psi(-z)\,dz = 0$. Then by the residue theorem we obtain

THEOREM 3. *For every integer $k \geqslant 2$, the following identity holds*:

$$(2)\qquad 2\sum_{n=1}^{\infty} n^{-k}\Psi(n) = k\zeta(k+1) - 2\gamma\zeta(k) - \sum_{l=2}^{k-1}\zeta(l)\zeta(k-l+1).$$

Here ζ denotes the Riemann zeta function and $\gamma = 0.577215664\cdots$ is Euler's constant.

PROOF. The determination of the residue of $z^{-k}\Psi^2(-z)$ at a positive integer is easily accomplished by means of the following well-known identities for the Ψ function.

$$\Psi(z) = \Psi(1-z) - \pi\cot\pi z, \qquad \Psi(1+z) = \Psi(z) + z^{-1}.$$

It is easily shown that

$$\lim_{z\to n} \frac{d}{dz}\{(z-n)^2 z^{-k}\Psi^2(-z)\} = (2-k)n^{-k-1} + 2n^{-k}\Psi(n).$$

Also, by the second identity noted above we have

$$z^{-k}\Psi^2(-z) = z^{-k}[z^{-1} + \Psi(1-z)]^2.$$

Since the representation

$$\Psi(1-z) = -\gamma - \sum_{n=2}^{\infty} \zeta(n) z^{n-1}$$

is valid for $|z| < 1$ (cf. [1]), the coefficient of z^{-1} in the expansion of $z^{-k}\Psi^2(-z)$ is

$$-2\zeta(k+1) + 2\gamma\zeta(k) + \sum_{l=2}^{k-1} \zeta(l)\zeta(k-l+1),$$

and this is the residue at 0. Q.E.D.

Next consider the functions $g_k(z) = z^{-k}\Psi(z)$. Proceeding as before we obtain the following formula for each odd integer $k \geqslant 3$:

$$2\sum_{n=1}^{\infty} n^{-k}\Psi(n) = -2\zeta(k+1) - 2\gamma\zeta(k) + \sum_{l=2}^{k-1} (-1)^l \zeta(l)\zeta(k-l+1). \tag{3}$$

Our method does not yield corresponding formulas for the even integers since the residues of $z^{-2p}\Psi(z)\Psi(-z)$ at $\pm n$ cancel one another.

Comparison of (2) and (3) for each odd integer $k = 2p - 1 \geqslant 3$ yields

$$(2p+1)\zeta(2p) = 2\sum_{l=1}^{p-1} \zeta(2l)\zeta(2p-2l). \tag{4}$$

Since

$$\zeta(2n) = (-1)^{n+1}(2\pi)^{2n} B_{2n}/2(2n)!,$$

we obtain the following relationship between the Bernoulli numbers due originally to Nörlund [4, p. 142]

$$(2p+1)B_{2p} = -\sum_{l=1}^{p-1} \binom{2p}{2l} B_{2l}B_{2p-2l}.$$

REMARK. The lack of a second expression for $\sum n^{-k}\Psi(n)$ when $k \geqslant 4$ is even is regrettable since such an expression would immediately yield relationships between the values of the Riemann zeta function at the even and the odd integers.

Finally, let

$$A_p = \sum_{l=2}^{2p+1} (-1)^l \zeta(l)\zeta(4p - l + 4).$$

Then for $k = 4p + 3$, equation (3) may be written as follows:

$$A_p = -\tfrac{1}{2}\zeta^2(2p + 2) + \gamma\zeta(4p + 3) + \zeta(4p + 4) + \sum_{n=1}^{\infty} n^{-4p-3}\Psi(n).$$

Hence

$$\sum_{n=1}^{\infty} (\zeta(2n) - \zeta(2n + 1)) = \lim_{p\to\infty} A_p = \frac{1}{2},$$

a result which can easily be derived by elementary methods.

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