# ITERATIVE SOLUTION OF LINEAR EQUATIONS IN BANACH SPACES 

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Abstract. Several results are obtained which generalize recent results of Kwon and Redheffer concerning the solution of $u-T u=f$ by means of iteration.

Throughout this note $E$ is a (real or complex) Banach space and $T$ is a continuous linear operator from $E$ to $E$. Let $f \in E$. In [1] Browder and Petryshyn considered the equation

$$
\begin{equation*}
u-T u=f \tag{1}
\end{equation*}
$$

and the convergence (to solutions $u$ of (1)) of the sequence $\left\{x_{n}\right\}$ defined by $x_{0} \in E$ and $x_{n+1}=T x_{n}+f$ for each $n=0,1,2, \ldots$, under the hypothesis that $\lim _{n \rightarrow \infty} T^{n} x$ exists for each $x \in E$. Kwon and Redheffer [4] considered equation (1) and the same sequence $\left\{x_{n}\right\}$ under the more general assumption that $\lim _{n \rightarrow \infty} T^{n} x$ exists for each $x$ in some subset of $E$. It is the purpose of this note to obtain generalizations of results in [4] by introducing a more general sequence $\left\{x_{n}\right\}$ (to be defined below) and obtaining results concerning $\left\{x_{n}\right\}$ analogous to those in [4].

Let $\left\{\lambda_{n}\right\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 \leqslant \lambda_{n} \leqslant 1$ for each $n=1,2, \ldots$, and $\lim _{n \rightarrow \infty} \lambda_{n}=0$. For notational simplicity we take $\lambda_{1}=1$. Define the infinite lower triangular matrix $A=\left[a_{n j}\right]$ by $a_{11}=\lambda_{1}=1, a_{i j}=0$ for each $j=2,3, \ldots, a_{n+1,1}=\lambda_{n+1} \quad$ for each $n=1,2, \ldots, a_{n+1, j+1}$ $=a_{n j}\left(1-a_{n+1,1}\right)$ for each $n=1,2, \ldots$, and each $j=1,2, \ldots, n$, and $a_{n+1, j}=0$ for each $n=1,2, \ldots$, and each $j=n+2, n+3, \ldots$

Remark 1. We note that $A=\left[a_{n j}\right]$ satisfies
(M1) $a_{n j} \geqslant 0$ for every $n$ and $j$, and $=0$ for $j>n$,
(M2) $\sum_{j=1}^{n} a_{n j}=1$ for every $n$, and
(M3) $\lim _{n \rightarrow \infty} a_{n j}=0$ for every $j$.
Hence $A$ is a regular matrix (cf. [3, p. 87]). Suppose $x_{0} \in E$ and let the sequence $\left\{x_{n}\right\}$ be defined recursively by

$$
\begin{equation*}
x_{n+1}=\left(1-\lambda_{n+1}\right)\left(T x_{n}+f\right)+\lambda_{n+1} x_{0} \tag{2}
\end{equation*}
$$

for each $n=0,1,2, \ldots$ This is a nonnormal Mann process which has been studied by Mann and Dotson.

[^0]Remark 2. If we take $\lambda_{n}=0$ for every $n$ then (2) yields the iterative process considered in [1] and [4] and the matrix $A$ reduces to the infinite identity matrix. If we take $\lambda_{n}=1 / n$ for every $n$ and $x_{0}=T y_{0}+f$ for $y_{0} \in E$ then (2) yields the iterative process studied by deFigueiredo and Karlovitz [2]. It is well known (by [2]) that this process often converges when the process in [1] and [4] does not.

Define the affine operator $S$ from $E$ to $E$ by $S x=T x+f$ for every $x \in E$. It is easily seen that

$$
\begin{equation*}
S^{j} x=T^{j} x+\sum_{i=1}^{j} T^{i-1} f \tag{3}
\end{equation*}
$$

for every $x \in E$ and for every $j=1,2, \ldots$ Moreover

$$
\begin{equation*}
x_{n+1}=\left(1-\lambda_{n+1}\right) S x_{n}+\lambda_{n+1} x_{0} \tag{4}
\end{equation*}
$$

for each $n=0,1,2, \ldots$. For each positive integer $n$, define polynomials $a_{n}(t)$ and $b_{n}(t)$ by

$$
\begin{equation*}
a_{n}(t)=\sum_{j=1}^{n} a_{n j} t^{j-1}, \quad b_{n}(t)=\frac{1-a_{n}(t)}{1-t} . \tag{5}
\end{equation*}
$$

Condition (M2) of Remark 1 insures that $b_{n}(t)$ is a polynomial. Using (5) together with condition (M2) we have

$$
b_{n}(t)=\sum_{j=1}^{n} \frac{a_{n j}\left(1-t^{j-1}\right)}{1-t}=\sum_{j=1}^{n} a_{n j} \sum_{i=1}^{j-1} t^{i-1}
$$

(We define $\sum_{i=1}^{0}=0$.) Now define linear operators $A_{n}$ and $B_{n}$ from $E$ to $E$ by $A_{n}=a_{n}(T)$ and $B_{n}=b_{n}(T)$ for each $n=1,2, \ldots$ By (5) we have

$$
\begin{equation*}
(I-T) B_{n}=B_{n}(I-T)=I-A_{n} \tag{6}
\end{equation*}
$$

(where $I$ denotes the identity operator on $E$ ).
Lemma 1. For each $n=1,2, \cdots$,

$$
\begin{equation*}
x_{n}=\sum_{j=1}^{n} a_{n j} S^{j-1} x_{0} \tag{7}
\end{equation*}
$$

Proof. The proof is by induction.

$$
\begin{aligned}
& x_{1}=\left(1-\lambda_{1}\right) S x_{0}+\lambda_{1} x_{0}=(1-1) S x_{0}+1\left(x_{0}\right)=x_{0}, \\
& x_{2}=\left(1-\lambda_{2}\right) S x_{1}+\lambda_{2} x_{0}=\left(1-\lambda_{2}\right) S x_{0}+\lambda_{2} x_{0} .
\end{aligned}
$$

If $x_{n}=\sum_{j=1}^{n} a_{n j} S^{j-1} x_{0}$ then

$$
\begin{equation*}
x_{n+1}=\left(1-\lambda_{n+1}\right) S x_{n}+\lambda_{n+1} x_{0}=\left(1-\lambda_{n+1}\right) \sum_{j=1}^{n} a_{n j} S^{j} x_{0}+\lambda_{n+1} x_{0} \tag{8}
\end{equation*}
$$

where we have used that $S$ is an affine operator. But since $\lambda_{n+1}=a_{n+1,1}$ and ( $\left.1-a_{n+1,1}\right) a_{n j}=a_{n+1, j+1}$ for each $j=1, \ldots, n$, equation (8) gives

$$
x_{n+1}=\sum_{j=1}^{n} a_{n+1, j+1} S^{j} x_{0}+a_{n+1,1} x_{0}=\sum_{j=1}^{n+1} a_{n+1, j} S^{j-1} x_{0}
$$

This completes the proof of Lemma 1. Combining equations (3) and (7) we obtain

Lemma 2. For every positive integer n,

$$
x_{n}=A_{n} x_{0}+B_{n} f
$$

Following [4] we define

$$
Q x=\lim _{n \rightarrow \infty} A_{n} x
$$

whenever this limit exists (unless otherwise stated all statements about limits refer to convergence in the norm). We also define

$$
D(Q)=\left\{x \in E: \lim _{n \rightarrow \infty} A_{n} x \text { exists }\right\}, \quad N(Q)=\left\{x \in E: \lim _{n \rightarrow \infty} A_{n} x=0\right\} .
$$

It is clear that $D(Q)$ is a linear subspace of $E$ and that the operator $Q$ is linear on $D(Q)$.

Remark 3. If $\lim _{n \rightarrow \infty} T^{n} x$ exists then, since $A$ is a regular matrix, $\lim _{n \rightarrow \infty} A_{n} x$ exists. We observe that in case $A$ is the Cesàro matrix $\left(\lambda_{n}=1 / n\right)$ various mean ergodic theorems give conditions under which $x \in D(Q)$.

Lemma 3. If $x \in D(Q)$ then $T x \in D(Q)$ and $Q(T x)=Q x$.
Proof. For any $n$ we have

$$
\begin{aligned}
A_{n}(T x) & =\sum_{j=1}^{n} a_{n j} T^{j} x \\
& =\sum_{j=1}^{n+1} a_{n+1, j} T^{j-1} x-a_{n+1,1} x+\sum_{j=1}^{n}\left(a_{n j}-a_{n+1, j+1}\right) T^{j} x .
\end{aligned}
$$

Using the property $a_{n+1, j+1}=a_{n j}\left(1-a_{n+1,1}\right)$ of $A$ and the linearity of $T$, we get

$$
\begin{align*}
A_{n}(T x) & =\sum_{j=1}^{n+1} a_{n+1, j} T^{j-1} x-a_{n+1,1} x+a_{n+1,1} \sum_{j=1}^{n} a_{n j} T^{j} x \\
& =\sum_{j=1}^{n+1} a_{n+1, j} T^{j-1} x-a_{n+1,1} x+a_{n+1,1} T\left(\sum_{j=1}^{n} a_{n j} T^{j-1} x\right)  \tag{9}\\
& =A_{n+1} x-a_{n+1,1} x+a_{n+1,1} T\left(A_{n} x\right) .
\end{align*}
$$

But $\lim _{n \rightarrow \infty} a_{n+1,1}=0$ and $\lim _{n \rightarrow \infty} A_{n+1} x=Q x$ since $x \in D(Q)$. Moreover $T$ is continuous so that $\lim _{n \rightarrow \infty} T\left(A_{n} x\right)=T Q x$. Thus the right-hand side of equation (9) converges to $Q x$ as $n \rightarrow \infty$. Hence we conclude that $\lim _{n \rightarrow \infty} A_{n}(T x)=Q x$. Therefore $T x \in D(Q)$ and $Q(T x)=Q x$ so the lemma is proved.

Remark 4. By Lemma 3, $x \in D(Q)$ implies

$$
\lim _{n \rightarrow \infty} A_{n} T x=Q x=\lim _{n \rightarrow \infty} A_{n} x
$$

so that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(A_{n} T x-A_{n} x\right)=0 \tag{10}
\end{equation*}
$$

for every $x \in D(Q)$. By Lemma 3, it is clear that $x \in D(Q)$ implies $T^{j} x \in D(Q)$ and $Q T^{j} x=Q x$ for every positive integer $j$. In addition $T$ commutes with $A_{n}$ for every $n$ and $T$ is continuous. Thus for $x \in D(Q)$,

$$
T(Q x)=Q(T x)=Q x
$$

This is easily extended to $T^{j} Q x=Q x$ for each positive integer $j$.
Theorem 1. Suppose there exists $y \in E$ such that $y-T y=f$. Then $\left\{x_{n}\right\}$ converges if and only if $x_{0}-y \in D(Q)$.

Proof. From Lemma 2

$$
\begin{equation*}
x_{n}=A_{n} x_{0}+B_{n} f \tag{11}
\end{equation*}
$$

for each $n=1,2, \ldots$. But $y-T y=f$ implies that

$$
B_{n} f=B_{n}(I-T) y \quad \text { for each } n=1,2, \ldots
$$

Thus by equation (6)

$$
B_{n} f=y-A_{n} y \quad \text { for each } n=1,2, \ldots
$$

Therefore (11) yields

$$
x_{n}=A_{n}\left(x_{0}-y\right)+y \quad \text { for each positive integer } n .
$$

Clearly $\left\{x_{n}\right\}$ has a limit if and only if $\left\{A_{n}\left(x_{0}-y\right)\right\}$ converges, i.e., if and only if $x_{0}-y \in D(Q)$.

This obviously generalizes Remark 1 of [4]. Before leaving Theorem 1, we note that if $\left\{x_{n}\right\}$ converges then the limit point is a solution of equation (1).

If $x \in E$ then $x$ is said to be an approximate solution of $u-T u=f$ provided that $x-T x-f \in N(Q)$. In the remainder of this note, $\rightarrow$ will denote convergence with respect to the weak topology on $E$.

Theorem 2. If $x_{0}$ is an approximate solution of $u-T u=f$, then every weak subsequential limit point $x$ of $\left\{x_{n}\right\}$ satisfies $x-T x=f$.

Proof. Let $x$ be a weak subsequential limit point of $\left\{x_{n}\right\}$. Then there is a subsequence $x_{n_{j}}=A_{n_{j}} x_{0}+B_{n_{j}} f$ such that $x_{n_{j}} \rightharpoonup x$.

Now $I-T$ is (strongly) continuous and linear and hence $I-T$ is weakly continuous. Thus

$$
\begin{equation*}
(I-T) x_{n_{j}} \rightharpoonup x-T x \tag{12}
\end{equation*}
$$

But

$$
\begin{equation*}
(I-T) x_{n_{j}}=(I-T) A_{n_{j}} x_{0}+(I-T) B_{n_{j}} f \tag{13}
\end{equation*}
$$

for each $j$.

Combining (12) and (13) we obtain

$$
A_{n_{j}} x_{0}-T A_{n_{j}} x_{0}+(I-T) B_{n_{j}} f \rightharpoonup x-T x
$$

By (6) (and the fact that $T$ commutes with $A_{n j}$ ) this becomes

$$
A_{n_{j}}\left(x_{0}-T x_{0}-f\right)+f \rightharpoonup x-T x
$$

Hence

$$
A_{n_{j}}\left(x_{0}-T x_{0}-f\right) \rightharpoonup x-T x-f
$$

But $x_{0}-T x_{0}-f \in N(Q)$ so that

$$
A_{n_{j}}\left(x_{0}-T x_{0}-f\right) \rightharpoonup 0 .
$$

The conclusion of the theorem follows immediately, since the weak topology is Hausdorff.
Theorem 2 clearly generalizes Remark 2 of [4].
Before proceeding to the next theorem we will need the following lemma, the proof of which is straightforward.

Lemma 4. If $x \in D(Q)$, then for each $n$,
(1) $A_{n} x, B_{n} x \in D(Q)$,
(2) $Q A_{n} x=Q x$, and
(3) $Q B_{n} x=\phi(n) Q x$ where

$$
\phi(n)=\left(\sum_{j=1}^{n} j a_{n j}\right)-1 \quad \text { and } \quad \lim _{n \rightarrow \infty} \phi(n)=+\infty
$$

Theorem 3. Suppose that $x_{0}, f \in D(Q)$ and that 0 is a weak subsequential limit point of $\left\{Q x_{n} / \phi(n)\right\}$. Then $x_{0}-T x_{0}-f \in N(Q)$.

Proof. We first note that $x_{n} \in D(Q)$ since by Lemma 4, $A_{n} x_{0}, B_{n} f$ $\in D(Q)$ and $D(Q)$ is a linear subspace of $E$. By hypothesis there is a subsequence $\left\{Q x_{n_{j}} / \phi\left(n_{j}\right)\right\}$ such that

$$
Q x_{n_{j}} / \phi\left(n_{j}\right) \rightharpoonup 0 \quad \text { as } j \rightarrow \infty
$$

For each $j, x_{n_{j}}=A_{n_{j}} x_{0}+B_{n_{j}} f$. Since $Q$ is linear on $D(Q)$ we have

$$
Q x_{n_{j}} / \phi\left(n_{j}\right)=Q A_{n_{j}} x_{0} / \phi\left(n_{j}\right)+Q B_{n_{j}} f / \phi\left(n_{j}\right) .
$$

Using (2) and (3) of Lemma 4 we get

$$
Q x_{n_{j}} / \phi\left(n_{j}\right)=Q x_{0} / \phi\left(n_{j}\right)+\phi\left(n_{j}\right) Q f / \phi\left(n_{j}\right)
$$

Therefore

$$
Q x_{n_{j}} / \phi\left(n_{j}\right)=Q x_{0} / \phi\left(n_{j}\right)+Q f
$$

Since $\lim _{j \rightarrow \infty} \phi\left(n_{j}\right)=+\infty,\left\{Q x_{n_{j}} / \phi\left(n_{j}\right)\right\}$ converges (strongly) to $Q f$. Hence $\left\{Q x_{n_{j}} / \phi\left(n_{j}\right)\right\}$ converges weakly to $Q f$ and since the weak topology on $E$ is

Hausdorff we conclude that $Q f=0$. Thus

$$
\lim _{n \rightarrow \infty} A_{n}\left(x_{0}-T x_{0}-f\right)=\lim _{n \rightarrow \infty}\left(A_{n} x_{0}-A_{n} T x_{0}\right)
$$

From equation (10)

$$
\lim _{n \rightarrow \infty}\left(A_{n} x_{0}-A_{n} T x_{0}\right)=0
$$

Therefore

$$
\lim _{n \rightarrow \infty} A_{n}\left(x_{0}-T x_{0}-f\right)=0,
$$

i.e., $x_{0}-T x_{0}-f \in N(Q)$.

Theorem 3 contains Remark 3 of [4] as a special case.
Theorem 4. Suppose there exists $u \in D(Q)$ such that $u-T u=f$. Then
(a) every solution of (1) is in $D(Q)$,
(b) $f \in N(Q)$, and
(c) $\left\{x_{n}\right\}$ converges if and only if $x_{0} \in D(Q)$.

Proof. Suppose $y \in E$ and $y-T y=f$. Since $u \in D(Q)$ and $u-T u=f$ then

$$
B_{n} f=B_{n}(I-T) u=\left(I-A_{n}\right) u=u-A_{n} u
$$

where we have used equation (6). But $u \in D(Q)$ implies that $\lim _{n \rightarrow \infty} A_{n} u$ $=Q u$. Therefore

$$
\begin{equation*}
\lim _{n \rightarrow \infty} B_{n} f=u-Q u \tag{14}
\end{equation*}
$$

Now $y-T y=f$ and $y-y=0 \in D(Q)$ so that by Theorem 1, the sequence $z_{n}=A_{n} y+B_{n} f$ converges (strongly) to some $z \in E$, i.e.,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(A_{n} y+B_{n} f\right)=z \tag{15}
\end{equation*}
$$

Equations (14) and (15) combine to imply that

$$
\lim _{n \rightarrow \infty} A_{n} y=z-u+Q u .
$$

Therefore $y \in D(Q)$.
(b) For each $n, A_{n} f=A_{n} u-A_{n} T u$ but $u \in D(Q)$ so by equation (10)

$$
\lim _{n \rightarrow \infty}\left(A_{n} u-A_{n} T u\right)=0 .
$$

Hence $\lim _{n \rightarrow \infty} A_{n} f=0$, i.e., $f \in N(Q)$.
(c) Since $u-T u=f$ we have by Theorem 1 that $\left\{x_{n}\right\}$ converges if and only if $x_{0}-u \in D(Q)$. But since $u \in D(Q), x_{0}-u \in D(Q)$ if and only if $x_{0}$ $\in D(Q)$. Theorem 4 clearly generalizes Remark 4 of [4].

Lemma 5. If $\left\{B_{n} f\right\}$ converges then

$$
\lim _{n \rightarrow \infty}\left(B_{n+1} f-T B_{n} f\right)=f
$$

Proof. For every $n$ we have

$$
\begin{aligned}
B_{n+1} f & =\sum_{j=2}^{n+1} a_{n+1, j} \sum_{i=1}^{j-1} T^{i-1} f\left(\text { recall that } \sum_{i=1}^{0}=0\right) \\
T B_{n} f & =\sum_{j=2}^{n} a_{n j} \sum_{i=1}^{j-1} T^{i} f .
\end{aligned}
$$

Subtraction yields

$$
B_{n+1} f-T B_{n} f=\left(\sum_{j=2}^{n+1} a_{n+1, j}\right) f+\sum_{j=2}^{n}\left(a_{n+1, j+1}-a_{n j}\right) \sum_{i=1}^{j-1} T^{i} f .
$$

By property (M2) of the matrix $A$ and the condition $a_{n+1, j+1}=a_{n j}\left(1-a_{n+1,1}\right)$ we have

$$
\begin{aligned}
B_{n+1} f-T B_{n} f & =\left(1-a_{n+1,1}\right) f-\sum_{j=2}^{n} a_{n+1,1} a_{n j} \sum_{i=1}^{j-1} T^{i} f \\
& =\left(1-a_{n+1,1}\right) f-a_{n+1,1} T B_{n} f .
\end{aligned}
$$

If $\left\{B_{n} f\right\}$ converges to $u$ then by the continuity of $T$,

$$
\lim _{n \rightarrow \infty} T B_{n} f=T u
$$

Therefore

$$
\lim _{n \rightarrow \infty}\left[\left(1-a_{n+1,1}\right) f-a_{n+1,1} T B_{n} f\right]=f
$$

since $\lim _{n \rightarrow \infty} a_{n+1,1}=0$. The conclusion of the lemma follows immediately.
Theorem 5. $\left\{B_{n} f\right\}$ converges if and only if there exists $u \in D(Q)$ such that $u-T u=f$.

Proof. If there exists $u \in D(Q)$ such that $u-T u=f$ then by equation (6)

$$
B_{n} f=B_{n}(I-T) u=\left(I-A_{n}\right) u=u-A_{n} u .
$$

Since $u \in D(Q)$, it is clear that $\left\{B_{n} f\right\}$ converges.
Conversely if $\left\{B_{n} f\right\}$ converges to $u$ then $\left\{T B_{n} f\right\}$ converges to $T u$. Therefore $\left\{B_{n+1} f-T B_{n} f\right\}$ converges to $u-T u$. But by Lemma 5, $\left\{B_{n+1} f-T B_{n} f\right\}$ converges to $f$. Hence $u-T u=f$. It remains only to show that $u \in D(Q)$. Clearly

$$
\lim _{n \rightarrow \infty}\left(A_{n}(0)+B_{n} f\right)=u
$$

so by Theorem $1,-u=0-u \in D(Q)$. Since $D(Q)$ is a linear subspace of $E$, it is clear that $u \in D(Q)$.

As a special case Theorem 5 contains Remark 5 of [4], since if $A=\left[a_{n j}\right]$ is the infinite identity matrix then

$$
b_{n}(t)=1-t^{n-1} / 1-t=1+t+\cdots+t^{n-2}
$$

and so $\left\{B_{n} f\right\}=\left\{f+T f+\cdots+T^{n-2} f\right\}$.

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