

## REMARKS ON SOME FIXED POINT THEOREMS

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**ABSTRACT.** A compact Hausdorff pseudo-topology is introduced on every closed convex bounded subset of a uniformly convex Banach space and is used to prove a previous theorem of the author.

In [7], we used a transfinite induction method which depends on the structure of the real line to prove the following fixed point theorem for multivalued nonexpansive mappings:

**THEOREM 1.** *Let  $K$  be a closed convex bounded nonempty subset of a uniformly convex Banach space and let  $T: K \rightarrow \mathcal{C}(K)$  be a nonexpansive mapping, where  $\mathcal{C}(K)$  denotes the family of nonempty compact (not necessarily convex) subsets of  $K$ , equipped with the Hausdorff metric. Then  $T$  has a fixed point, i.e. there exists  $x \in K$  such that  $x \in Tx$ .*

The properties of real numbers we used are the order property and the separability, or more explicitly, that a decreasing nonnegative transfinite sequence indexed by ordinals less than the uncountable ordinal  $\Omega$  must be eventually constant.

Recently, Caristi and Kirk [1], [2], [5] have proven the following theorem and have given several interesting applications:

**THEOREM 2** [1], [2]. *Let  $X$  be a complete metric space, and let  $g: X \rightarrow X$  be a self-map of  $X$ . Suppose that there exists a lower semicontinuous nonnegative real-valued mapping  $\xi$  such that for all  $x$  in  $X$ ,*

$$(1) \quad d(x, g(x)) \leq \xi(x) - \xi(g(x)).$$

*Then  $g$  has a fixed point.*

Chi Song Wong [8] has given a simple proof of the Caristi-Kirk theorem by using the transfinite induction method mentioned above. On the other hand, we define a pseudo-compact-Hausdorff-topology on any closed convex bounded subset of a uniformly convex Banach space and give Theorem 1 a simpler and more conceptual proof. It is our feeling that the existence of such a compact Hausdorff pseudo-topology may well serve to explain the similarity between uniform convexity and compactness in some aspects of geometric fixed point theory.

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Let  $(X, d)$  be a metric space. A sequence  $\{x_n\}$  in  $X$  is said to  $\Delta$ -converge to a point  $x \in X$ , written  $x_n \rightarrow^\Delta x$ , if

$$(2) \quad \limsup_i d(x_{n_i}, x) \leq \limsup_i d(x_{n_i}, y)$$

for every subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  and every  $y \in X$ . In the terminology of asymptotic center (Edelstein [3], Lim [6]), this says that  $x$  is an asymptotic center of every subsequence of  $\{x_n\}$ .  $\{x_n\}$  is said to  $\Delta$ -converge strongly to  $x$  if

$$(3) \quad \lim d(x_n, x) \leq \liminf d(x_n, y) \quad \text{for every } y \in X.$$

This is equivalent to saying that all subsequences of  $\{x_n\}$  have a common asymptotic center ( $= x$ ) and asymptotic radius ( $= \lim d(x_n, x)$ ). In general,  $x$  is not unique. That strong  $\Delta$ -convergence implies  $\Delta$ -convergence is obvious.

$\Delta$ -convergence has the following properties of which only the first two are satisfied by strong  $\Delta$ -convergence:

- (i) if  $x_n = x$  for every  $n$ , then  $x_n \rightarrow^\Delta x$ ;
- (ii) if  $x_n \rightarrow^\Delta x$ , then every subsequence of  $\{x_n\}$   $\Delta$ -converges to  $x$ ;
- (iii) if  $\{x_n\}$  does not  $\Delta$ -converge to  $x$ , then there exists a subsequence of which every subsequence does not  $\Delta$ -converge to  $x$ .

Clearly these definitions and properties can be formulated for nets. A set  $X$  equipped with a convergence class satisfying (i), (ii) and (iii) (or only (i) and (ii)) will be called a pseudo-topological space and the convergence class will be called a pseudo-topology on  $X$ . Note that we need only one additional axiom to define topological spaces by convergence classes, see e.g., Kelley [4]. By using nets, concepts in topological spaces can be carried over to quasi-topological spaces. Thus a quasi-topological space is compact if every net in it has a convergent subnet and is Hausdorff if a net can converge to at most one point. In what follows, (strong)  $\Delta$ -topology will refer to the quasi-topology with the convergence class given by sequences satisfying (2) ((3) respectively). One may replace "sequences" by "nets" in this definition and will obtain the same conclusions of Theorems 3 and 4 below.

A metric space is said to be  $\Delta$ -complete if for every bounded sequence (or net)  $\{x_n\}$  in  $X$  there is an  $x \in X$  such that

$$\limsup_n d(x_n, x) \leq \limsup_n d(x_n, y)$$

for every  $y \in X$  i.e.  $\{x_n\}$  has an asymptotic center in  $X$ .

**THEOREM 3.** *Every bounded  $\Delta$ -complete metric space  $X$  is strongly  $\Delta$ -compact (and hence  $\Delta$ -compact), i.e. every sequence in  $X$  has a strongly  $\Delta$ -convergent subsequence.*

**THEOREM 4.** *Every closed convex bounded subset of a uniformly convex Banach space is compact Hausdorff under the (strong)  $\Delta$ -topology.*

Our proofs of Theorems 3 and 4 need the following set-theoretical result.

For two sequences  $\{x_n\}$  and  $\{y_n\}$  in a set, let us say  $\{x_n\}$  is an essential

subsequence of  $\{y_n\}$  if there exists a positive integer  $N$  such that  $\{x_n\}_{n \geq N}$  is a subsequence of  $\{y_n\}$ .

**PROPOSITION 1.** *Let  $X$  be a set and let  $\{x_n\}$  be a sequence in  $X$ . Let  $r$  be a real-valued function whose domain is the set of subsequences of  $\{x_n\}$ . Suppose  $r(y) \leq r(z)$  whenever  $y$  is an essential subsequence of  $z$ . Then there is a subsequence  $w$  of  $\{x_n\}$  such that  $r(z) = r(w)$  for every subsequence  $z$  of  $w$ .*

**PROOF.** Denote by  $\mathfrak{F}$  the family of subsequences of  $\{x_n\}$ . Define an ordering  $\leq$  on  $\mathfrak{F}$  as follows:

For  $x, y \in \mathfrak{F}$ , we put  $x < y$  if  $x$  is an essential subsequence of  $y$  and  $r(x) < r(y)$ . Then we say  $x \leq y$  if  $x < y$  or  $x$  is identically equal to  $y$ .

It is easy to check that  $\leq$  is a reflexive, antisymmetric and transitive relation. Let  $\mathcal{C}$  be a chain in  $\mathfrak{F}$ . Let  $r = \inf\{r(x) : x \in \mathcal{C}\}$ . If there is an  $x \in \mathcal{C}$  such that  $r(x) = r$ , then  $x$  is a lower bound for  $\mathcal{C}$ . Therefore we assume that such  $x$  does not exist. Let  $x_n$  be a sequence in  $\mathcal{C}$  such that  $r(x_n)$  strictly decreases to  $r$ . Since  $\mathcal{C}$  is a chain, we must have  $x_1 > x_2 > \dots$ . By using the diagonal process, dropping a finite number of terms in each sequence  $X_n$  if necessary, we obtain a sequence  $y$  which is an essential subsequence of  $x_n$  for every  $n$ . Then, by assumption,  $r(y) < r(x_n)$  for every  $n$  and, hence,  $r(y) < r(x)$  for every  $x \in \mathcal{C}$ . Since each  $x \in \mathcal{C}$  is an essential subsequence of  $x_n$  for some  $n$ , we conclude that  $y$  is a lower bound for  $\mathcal{C}$ .

By Zorn's lemma,  $\mathfrak{F}$  has a minimal element  $z$ . Let  $w$  be a subsequence of  $z$ . Then  $r(w) \leq r(z)$ . If  $r(w) < r(z)$ , then  $w < z$  and by the minimality and antisymmetry we must have  $w = z$  which implies  $r(w) = r(z)$ , a contradiction. Hence  $r(w) = r(z)$ . **Q.E.D.**

**PROOF OF THEOREM 3.** Let  $\{x_n\}$  be a sequence in  $X$ . For every subsequence  $\{x_{n_i}\}$ , let

$$r(\{x_{n_i}\}) = \inf \left\{ \limsup_i d(x_{n_i}, y) : y \in X \right\}.$$

By Proposition 1  $\{x_n\}$  contains a subsequence which we still denote by  $\{x_n\}$  such that

$$r(\{x_n\}) = r(\{x_{n_i}\}) = r$$

for every subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ . By  $\Delta$ -completeness, there exists an  $x \in X$  such that

$$\limsup d(x_n, x) = r(\{x_n\}).$$

For every subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ , we have

$$\begin{aligned} \limsup d(x_{n_i}, x) &\leq \limsup d(x_n, x) \\ &= r(\{x_n\}) = r(\{x_{n_i}\}) \leq \limsup d(x_{n_i}, x); \end{aligned}$$

thus

$$\limsup d(x_{n_i}, x) = r(\{x_{n_i}\}).$$

This shows that all subsequences  $\{x_{n_i}\}$  of  $\{x_n\}$  have a same asymptotic center  $x$  and a same asymptotic radius  $r$ . Q.E.D.

PROOF OF THEOREM 4. This follows from Theorem 3 and the uniqueness of asymptotic center as proved by Edelstein [3]. Q.E.D.

Let us now give

PROOF OF THEOREM 1. By a standard argument, there exist sequences  $\{x_n\}$  and  $\{y_n\}$  such that  $y_n \in Tx_n$  and  $\|x_n - y_n\| \rightarrow 0$ . By Theorem 4,  $\{x_n\}$  has a  $\Delta$ -convergent subsequence which we still denote by  $\{x_n\}$ . Let  $x$  be its  $\Delta$ -limit. We assert that  $x \in Tx$ . For each  $n$ , choose  $p_n \in Tx$  such that  $\|p_n - y_n\| \leq \|x - x_n\|$ . Since  $Tx$  is compact, there exists a convergent subsequence  $\{p_{n_i}\}$  of  $\{p_n\}$  such that  $p_{n_i} \rightarrow p$  for some  $p \in Tx$ . It can be easily shown, by using  $\|x_n - y_n\| \rightarrow 0$ ,  $\|p_n - y_n\| \leq \|x - x_n\|$  and  $x_n \rightarrow^\Delta x$ , that  $x_{n_i} \rightarrow^\Delta p$ . Since also  $x_{n_i} \rightarrow^\Delta x$ , we must have  $x = p \in Tx$  by the uniqueness of  $\Delta$ -limit. Q.E.D.

REMARK. I am indebted to the referee for informing me that K. Goebel has discovered independently a similar proof of Theorem 1 in a paper published in Ann. Univ. Mariae Curie-Skłodowska.

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