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THE UNDECIDABILITY OF A FUNDAMENTAL PROBLEM IN CLUSTER SET THEORY

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ABSTRACT. The undecidability of the existence of a nonmetrizable normal picket fence space is established and used to establish the undecidability of the following statement: Any family of approach curves (approaching a point in $\mathbf{R}^2$) along which cluster sets can be arbitrarily preassigned has a nonintersecting truncation.

1. Let Γ denote the set of all curves in $\mathbf{R}^2$ that approach the origin, and for $\gamma \in \Gamma$ and f: $\mathbf{R}^2 \to \mathbf{R}$, let $C(f, \gamma)$ denote the cluster set of f at 0 along the curve γ (i.e., $C(f, \gamma) = \{\lambda$: there exists a sequence $\{P_n\}$ of distinct points on γ such that $P_n \to 0$ and $f(P_n) \to \lambda\}$). A search in [2] for large subsets Γ_0 of Γ along which cluster sets could be arbitrarily preassigned led to the following conjecture:

(T) A subfamily Γ_0 of Γ has the property that for any given collection $\{C(\gamma)$: $\gamma \in \Gamma_0\}$ of closed sets there exists a function f such that $C(f, \gamma) = C(\gamma)$ for every γ in Γ_0 if and only if Γ_0 has a nonintersecting truncation.

In [3] statement (T) was shown to be equivalent to the restriction of the normal Moore space conjecture to a special class of spaces. Consequently (see [3]), (T) is consistent with the usual axioms of set theory (ZFC). The purpose of this paper is to show that the negation of (T) is also consistent. This will be done by establishing the existence of a nonmetrizable normal picket fence space. Such a space will then be transformed into a counterexample to (T). Martin's Axiom plus the negation of the continuum hypothesis (MA $+ \neg$ CH) will be assumed (cf. [8]).

2. **A nonmetrizable normal picket fence space.** As in [3], for $S \subset \mathbf{R}$ and m: $S \to \mathbf{R}$, let (S, m) denote the family of half-lines $y = m(s)x + s$, $x \geqslant 0$, $s \in S$, and let $X(S, m)$ denote the associated picket fence space (the points of $X(S, m)$ are the points of the right half-plane $x > 0$ together with the points of S; the right half-plane has the discrete topology and basic neighborhoods of points s of S are the truncated lines $y = m(s)x + s$, $0 \leqslant x < \tau$, $\tau > 0$).

The main difference between these spaces and the familiar tangent disk

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spaces (see [9]) is that these spaces generally do not have the countable cellularity property on the discrete subspace S (for an extreme case, see [4]). Hence, the usual method of obtaining a normal noncollectionwise Hausdorff space from a Q-set does not apply to picket fence spaces. Also, it is not at all clear how a nonmetrizable normal space formed from triangular disks or wedges could be converted into one formed from pickets. Thus, it appears that a special approach like the one given below is necessary.

THEOREM (MA + $\neg$ CH). *There exists a nonmetrizable normal picket fence space.*

PROOF. Let S be any set of real numbers of power $\aleph_1$. By [7, pp. 38–40] and [5, p. 154], MA + $\neg$ CH implies that S is a Q-set and has Lebesgue measure zero. Note that $X(S, m)$ is normal no matter how m is defined (cf. [1, Example E]).

Let ϕ be a function from $\mathbf{R}$ to $\mathbf{R}$ whose derivative is equal to $-\infty$ at every point of S. This is possible because S has measure zero (see [6, p. 214]). Let m be the restriction of ϕ to S. Observe that if $X(S, m)$ were collectionwise Hausdorff, then for some uncountable subset A of S and some positive number ε, the family (A, m) would be nonintersecting on the strip $0 < x < \varepsilon$. The following argument shows that this is not the case.

Let A be an uncountable subset of S and let a be a point of A which is also a limit point of A. Since $\phi'(a) = -\infty$, there exists a sequence $\{c_n\}$ in A such that $(\phi(c_n) - \phi(a))/(c_n - a) \leqslant -n$ for every n. Hence, there exist sequences $\{a_n\}$ and $\{b_n\}$ in A such that $0 < (b_n - a_n)/(m(a_n) - m(b_n)) < 1/n$ for every n. In other words, the family (A, m) intersects itself on every strip of the form $0 < x < 1/n$.

Since metrizability and collectionwise normality are equivalent for Moore spaces [1], the theorem follows.

3. **A counterexample to (T).** In [3] it was shown that statement (T) is equivalent to the statement: *Every normal Moore space $X(\Gamma_0)$ is metrizable.* (The points of $X(\Gamma_0)$ are the points of the punctured plane $\mathbf{R}^2 - \{0\}$ together with the curves of Γ_0; the punctured plane has the discrete topology and basic neighborhoods of a point γ in Γ_0 are the truncations of γ.)

THEOREM (MA + $\neg$ CH). *There exists a nonmetrizable normal Moore space $X(\Gamma_0)$.*

PROOF. Let $X(S, m)$ be a nonmetrizable normal picket fence space and let Γ_0 be the family of parabolic approaches $y = m(s)x^2 + sx$, $x > 0$, $s \in S$. Obviously, $X(\Gamma_0)$ is a Moore space. Normality and noncollectionwise normality follow from the fact that the truncated lines $y = m(s)x + s$, $0 < x < \sigma$, and $y = m(t)x + t$, $0 < x < \tau$, intersect if and only if the truncated parabolas $y = m(s)x^2 + sx$, $0 < x < \sigma$, and $y = m(t)x^2 + tx$, $0 < x < \tau$, intersect.

REMARK. In the above theorems, the full strength of MA + $\neg$ CH is not needed; all that is needed is the existence of a Q-set (cf., e.g., Theorem 5(b) of [7]).

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