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CONTINUOUS ACTIONS OF COMPACT LIE GROUPS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. M. H. A. Newman proved that if M is a connected topological manifold with metric d, there exists a number $\varepsilon > 0$, depending only upon M and d, such that every compact Lie group acting effectively on M has at least one orbit of diameter at least ε. In this paper the authors consider the case where M is a Riemannian manifold and d is the distance function on M arising from the Riemannian metric. They obtain estimates for ε in terms of convexity and curvature invariants of M.

1. **Introduction.** In 1931 M. H. A. Newman proved the following result.

THEOREM (NEWMAN [8]). *If M is a connected topological manifold with metric d, there exists a number $\varepsilon = \varepsilon(M,d) > 0$, depending only upon M and d, such that every compact Lie group G acting effectively on M has at least one orbit of diameter at least ε.*

Recently several investigators, including ourselves [4], [6], [7], have studied compact groups of *isometries* on a Riemannian manifold M and have obtained estimates for ε in terms of convexity and curvature invariants of M. In this paper we consider *continuous* actions of compact Lie groups on a Riemannian manifold M and we obtain results which compare quite favorably with the results for isometric actions. Moreover, our arguments are surprisingly simple. Specifically, we have obtained the following results: We call a subset S of M *convex* if for every pair of points in S there exists a unique distance measuring geodesic in S joining them. For $x \in M$, the *radius of convexity of M at x*, which we denote by r_x, is defined as the supremum of the radii of all convex embedded open balls centered at x.

THEOREM 1. *Let M be a Riemannian manifold with nonpositive sectional curvature. Let*

$$\bar{r} = \sup_{x \in M} r_x$$

and suppose r is any number, $0 < r < \bar{r}$. If G is any compact Lie group acting

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continuously and effectively on M, then there exists at least one orbit of diameter at least $r/2$. *In particular if* $\bar{r} = +\infty$, *there exist orbits of arbitrarily large diameters.*

The technique of proof depends upon the original approach of Newman as presented in a paper by Andreas Dress [2] together with the well-known fact that the exponential map locally stretches distances for manifolds of nonpositive curvature. For manifolds of bounded curvature $K \leqslant b^2$, we obtain an analogous result by using a technique similar to that of Hoffman [4] which compares distances locally on M with distances on a space form of constant positive curvature b^2.

THEOREM 2. *Let M be a Riemannian manifold with curvature bounded above by a positive constant* b^2. *Suppose r is any number,* $0 < r < \min\{\pi/2 \cdot b^{-1}, \bar{r}\}$. *If G is any compact Lie group acting continuously and effectively on M, then there exists at least one orbit of diameter at least* $2r/(\pi + 2)$.

As for a best possible type result, we have the following candidate.

THEOREM 3. *Let G be a compact Lie group acting continuously and effectively on a compact orientable Riemannian manifold M. Then there exists at least one orbit which is not contained in any open convex subset of M.*

In [9], P. A. Smith gave a generalization of Newman's Theorem to cohomology manifolds. In this paper Smith claims [9, p. 448] that if a compact Lie group acts continuously and effectively on an n-sphere S^n (with the standard metric) there exists at least one orbit which is not contained in any open hemisphere. Theorem 3 is a natural generalization of Smith's claim. The existence of such a result was suggested by the techniques of G. Bredon in [1, III.9] where he gives a proof of Smith's version of the Newman Theorem. The assumptions of compactness and orientability appear to be needed only for technical reasons.

2. **Proof of Theorems 1 and 2.** We will use the following two results:

A. LEMMA (A. DRESS [2]). *Let U be an open, relatively compact and connected subset of* R^n. *If* Z_p *acts continuously, effectively and invariantly on* $\bar{U}$, *then*

$$D = \operatorname{Max}\{\operatorname{Min}\{\|x - y\| \,|\, y \in \partial\bar{U}\} | x \in U\}$$
$$\leqslant C = \{\operatorname{Max} \|x - \tau x\| \,|\, \tau \in Z_p, x \in \partial\bar{U}\}.$$

Here $\|x - y\|$ *is the euclidean norm in* R^n.

B. PROPOSITION. *Suppose* $K \leqslant b^2$ (*respectively* $K \leqslant 0$) *on a Riemannian manifold M with distance function d. Let* $B_r(z) = \{y | d(y,z) < r\}$ *be a convex embedded ball centered at z in M. Suppose further that* $r < \pi b^{-1}/2$ (*respectively* $0 < r < \infty$ *when* $K \leqslant 0$). *For any* $x, y \in B_r(z)$, *if* $\hat{x} = \exp_z^{-1} x$ *and* $\hat{y} = \exp_z^{-1} y$, *then* $d(x,y) \geqslant (2/\pi)\|\hat{x} - \hat{y}\|$ (*respectively* $d(x,y) \geqslant \|\hat{x} - \hat{y}\|$ *when* $K \leqslant 0$). *Here* $\|x - y\|$ *is the euclidean norm in the tangent space* M_z.

Lemma A appears as Lemma 3 in [2]. Proposition B, for the case $K \leqslant 0$, is a well-known fact. The proof of Proposition B for Riemannian manifolds with curvature bounded above by a positive constant will be presented in the next section.

We prove Theorems 1 and 2 simultaneously. Without loss of generality, we may assume $G = Z_p$. Fix any $z \in M$ and let $r_z =$ the radius of convexity at z. For any $r > 0$ satisfying

$$r < \begin{cases} r_z & \text{if } K \leqslant 0, \\ \operatorname{Min}\{r_z, \pi/2 \cdot b^{-1}\} & \text{if } K \leqslant b^2, \end{cases}$$

and any α, $0 < \alpha < 1$, suppose that

$$\text{(H)} \qquad d(x, \tau x) < (1-\alpha)r \quad \text{for all } x \in M, \text{ all } \tau \in Z_p.$$

Define $U = \bigcup_{\tau \in Z_p} \tau B_{\alpha r}(z)$. By construction, $\overline{U}$ is Z_p-invariant. Furthermore, by (H), $\overline{B}_{\alpha r}(z) \subset \overline{U} \subset \overline{B}_r(z)$.

Now lift the action of Z_p on $\overline{U}$ to an action of Z_p on the closed set $\exp_z^{-1} \overline{U}$, i.e. let $\tau \in Z_p$ act on $\exp_z^{-1} \overline{U}$ by $\exp_z^{-1} \circ \tau \circ \exp_z$. For convenience, we will let $U_\wedge = \exp_z^{-1} U$. Clearly $\overline{U}_\wedge$ is Z_p invariant and

$$\{\hat{x} \in M_z \mid \|\hat{x}\| \leqslant \alpha r\} = \exp_z^{-1} \overline{B}_{\alpha r}(z) \subset \overline{U}_\wedge$$
$$\subset \exp_z^{-1} \overline{B}_r(z) = \{\hat{x} \in M_z \mid \|\hat{x}\| \leqslant r\}.$$

The left-hand inclusion implies

$$D = \operatorname{Max}\{\operatorname{Min}\{\|\hat{x} - \hat{y}\| \mid \hat{y} \in \partial \overline{U}_\wedge\} \mid \hat{x} \in U_\wedge\} \geqslant \alpha r.$$

(Simply let $\hat{x} = 0$.) Since $\overline{B}_r(z)$ is a convex, embedded ball with $r < \pi/2 \cdot b^{-1}$ when $K \leqslant b^2$ ($r < \infty$ when $K \leqslant 0$), we may apply Proposition B, which, together with (H) implies

$$C = \operatorname{Max}\{\|\hat{x} - \tau\hat{x}\| \mid \hat{x} \in \partial \overline{U}_\wedge, \tau \in Z_p\} < \begin{cases} (1-\alpha)r & \text{if } K \leqslant 0, \\ (1-\alpha)\pi r/2 & \text{if } K \leqslant b^2. \end{cases}$$

But, by Lemma A, $D \leqslant C$. This implies

$$\alpha r < \begin{cases} (1-\alpha)r & \text{if } K \leqslant 0, \\ (1-\alpha)\pi r/2 & \text{if } K \leqslant b^2, \end{cases}$$

or

$$\alpha < \begin{cases} 1/2 & \text{if } K \leqslant 0, \\ \pi/\pi + 2 & \text{if } K \leqslant b^2. \end{cases}$$

Therefore, (H) is *false* for

$$\alpha = \begin{cases} 1/2 & \text{if } K \leqslant 0, \\ \pi/\pi + 2 & \text{if } K \leqslant b^2; \end{cases}$$

i.e. there exists an $x \in M$ whose orbit has diameter at least $r/2$ if $K \leqslant 0$; $2r/\pi + 2$ if $K \leqslant b^2$. This completes the proof of Theorems 1 and 2.

REMARK. For Z_2 actions, Lemma A may be strengthened to say that $2D \leqslant C$ (there is a misprint on p. 206 of [2]). Moreover if G is not a finite group of odd order, it must contain an involution. Using these facts in the above argument shows that:

COROLLARY. *Let M be a Riemannian manifold with sectional curvature $K \leqslant b^2$ (respectively $K \leqslant 0$). Suppose r satisfies $0 < r < \operatorname{Min}\{\pi/2 \cdot b^{-1}, \bar{r}\}$ (respectively $0 < r < \bar{r}$ when $K \leqslant 0$). If G is a compact Lie group acting continuously and effectively on M and if G is not a finite group of odd order, then there exists at least one orbit of diameter at least $4r/\pi + 4$ (respectively $2r/3$ when $K \leqslant 0$).*

3. **A distance preserving property of the exponential map.** In this section we outline a proof of Proposition B in the case where the curvature of M is bounded above by a positive constant b^2.

Let $z \in M$ and choose any r, $0 < r < \operatorname{Min}(r_z, \pi/2 \cdot b^{-1})$. For any $x, y \in B_r(z)$, we must show $d(x,y) \geqslant (2/\pi)\|\hat{x} - \hat{y}\|$.

We proceed as follows. Let $S^n(b^{-1})$ be the n-sphere of constant curvature b^2, and fix $p \in S^n(b^{-1})$.

The choice of r ensures that $B_r(z)$ and $\tilde{B}_r(p)$, the geodesic ball of radius r centered at $p \in S^n(b^{-1})$, are both convex. Identify M_z with $S^n(b^{-1})_p$ by a linear isometry i. We have the bijection

$$B_r(z) \xrightarrow{\exp_z^{-1}} M_z \xrightarrow{i} S^n(b^{-1})_p \xrightarrow{\widetilde{\exp}_p} \tilde{B}_r(p).$$

Here, $\widetilde{\exp}_p$ is the exponential map of $S^n(b^{-1})$ restricted to the tangent space at p. Let $\eta = \widetilde{\exp}_p \circ i \circ \exp_z^{-1}$.

Since $x, y \in B_r(z)$, neither x nor y is conjugate to z along any geodesic in $B_r(z)$. Similarly, $\eta x, \eta y \in \tilde{B}_r(p)$ are not conjugate along any geodesic in $\tilde{B}_r(p)$. Therefore we may apply the Rauch Comparison Theorem [5, p. 76] to conclude:

If γ is a length-measuring geodesic connecting x to y in $B_r(z)$,

$$\tilde{d}(\eta x, \eta y) \leqslant \text{Length}(\eta \circ \gamma) \leqslant \text{Length}(\gamma) = d(x,y). \tag{1}$$

Here $\tilde{d}$ is the distance function in $S^n(b^{-1})$. Now suppose δ is a length-measuring geodesic joining ηx to ηy in $\tilde{B}_r(p) \subset S^n(b^{-1})$. The curve $\widetilde{\exp}_p^{-1} \circ \delta$ connects $\widetilde{\exp}_p^{-1} \eta x$ to $\widetilde{\exp}_p^{-1} \eta y$, so

$$\|\widetilde{\exp}_p^{-1} \eta x - \widetilde{\exp}_p^{-1} \eta y\| \leqslant \text{Length}(\widetilde{\exp}_p^{-1} \circ \delta).$$

Here length is measured in $S^n(b^{-1})_p$. Since i is an isometry (and $i\hat{x}$

$= \widetilde{\exp}_p^{-1} \eta x$, etc.),

$$\|x - y\| \leqslant \text{Length } (\widetilde{\exp}_p^{-1} \circ \delta). \tag{2}$$

We now wish to compare the length of $\widetilde{\exp}_p^{-1} \circ \delta$ with the length of $\delta \subset S^n(b^{-1})$. A straightforward computation, using polar coordinates to express $\widetilde{\exp}_p$ and the arc-length formula, yields

$$((\sin br)/br) \text{ Length } (\exp^{-1} \circ \delta) \leqslant \text{Length } \delta.$$

Since $r < \pi/2 \cdot b^{-1}$ and δ measures length,

$$2/\pi \cdot \text{Length } (\exp_p^{-1} \delta) \leqslant \tilde{d}(\eta x, \eta y). \tag{3}$$

Combining (1), (2) and (3),

$$2\|\hat{x} - \hat{y}\|/\pi \leqslant d(x,y),$$

which is the desired inequality.

4. **Proof of Theorem 3.** In [1, pp. 154–156] Bredon proves a version of Newman's Theorem due to P. A. Smith. As a preliminary he gives a result, Theorem 9.3 of [1], where he proves:

If M is a compact orientable manifold and $\mathfrak{U}$ is any open covering of M such that $H^n(K(\mathfrak{U}), Z) \to \check{H}^n(M, Z)$ is onto, then there *does not* exist a continuous effective action of a compact Lie group on M such that each orbit is contained in some member of $\mathfrak{U}$.

Here, $H(K(\mathfrak{U}), Z)$ is the integral cohomology of $K(\mathfrak{U})$, the nerve of the covering $\mathfrak{U}$, and $\check{H}(M, Z)$ is the integral Čech cohomology. The map is the natural induced map.

So now let $\mathfrak{U}$ be the covering consisting of all open, convex sets of the Riemannian manifold M.

Since the intersection of convex sets is a convex set and convex sets are contractible, $H^q(|\sigma|, Z) = 0$ for all $\sigma \in K(\mathfrak{U})$, $q \geqslant 1$. By Leray's Theorem [3, p. 44], the induced map $H^q(K(\mathfrak{U}), Z) \to \check{H}^q(M, Z)$ is an isomorphism for all $q \geqslant 0$. In particular, $H^n(K(\mathfrak{U}), Z) \to \check{H}^n(M, Z)$ is onto. Thus, Bredon's result is applicable to the covering by convex sets, proving Theorem 3.

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