PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

A GENERALIZATION OF ANDERSON'S THEOREM ON UNIMODAL FUNCTIONS

SOMESH DAS GUPTA[1]

ABSTRACT. Anderson (1955) gave a definition of a unimodal function on R^n and obtained an inequality for integrals of a symmetric unimodal function over translates of a symmetric convex set. Anderson's assumptions, especially the role of unimodality, are critically examined and generalizations of his inequality are obtained in different directions. It is shown that a marginal function of a unimodal function (even if it is symmetric) need not be unimodal.

1. **Introduction**. A function $f\colon R^n \equiv [0, \infty)$ is said to be unimodal by Anderson (1955) if

$$D(u) \equiv \{x: f(x) \geqslant u\} \tag{1.1}$$

is convex for all u, $0 < u < \infty$. The main result of this paper is a generalization of the following theorem of Anderson (1955) on the integrals of a symmetric unimodal function over translates of a symmetric convex set.

THEOREM (ANDERSON). *Let E be a symmetric (i.e., $E = -E$) convex set in R^n and f be a function on R^n to $[0, \infty)$ such that f is symmetric (i.e., $f(x) = f(-x)$), unimodal, and $\int_E f(x)\,\mu_n(dx) < \infty$, where μ_n is the Lebesgue measure on R^n. Then for any fixed $y \in R^n$ and $0 \leqslant \lambda \leqslant 1$*

$$\int_E f(x + \lambda y)\,\mu_n(dx) \geqslant \int_E f(x + y)\,\mu_n(dx). \tag{1.2}$$

This result was extended by Mudholkar (1966) by replacing the condition of symmetry with the condition of invariance under a linear Lebesgue measure-preserving group G of transformations of R^n onto R^n.

THEOREM (MUDHOLKAR). *Let E be a convex, G-invariant set in R^n and f be a function on R^n to $[0, \infty)$ such that f is G-invariant unimodal and $\int_E f(x)\,\mu_n(dx) < \infty$. Then for fixed $y \in R^n$ and any y^* in the convex hull of the G-orbit of $\{y\}$*

Received by the editors July 2, 1974.

AMS (MOS) subject classifications (1970). Primary 26A69, 26A87, 52A40.

Key words and phrases. Unimodal function, convex set, invariance, marginal function, inequalities.

[1]This work was supported in part by U. S. Army Research Grant DA-ARO-D-31-124-70-G-102 at the University of Minnesota and in part by National Science Foundation Grant GS-39906 at the Institute for Mathematical Studies in the Social Sciences, Stanford University. The author wishes to thank Professor T. W. Anderson for some useful discussions.

$$\int_E f(x+y^*)\,\mu_n(dx) \geqslant \int_E f(x+y)\,\mu_n(dx). \tag{1.3}$$

Note that Anderson's theorem follows from Mudholkar's by taking G to be the group of sign-change transformations.

Let us consider Anderson's theorem again and define

$$h(y) \equiv \int_E f(x+y)\,\mu_n(dx) \tag{1.4}$$

$$= \int f(x+y) I_{E\times R^n}(x,y)\,\mu_n(dx), \tag{1.5}$$

where I is the indicator function. It is shown in later sections that the conclusions of Anderson's theorem, i.e.,

$$h(y) = h(-y), \quad h(\lambda y) \geqslant h(y), \qquad 0 \leqslant \lambda \leqslant 1, \tag{1.6}$$

still hold, if $h(y)$ is defined by

$$h(y) = \int_{R^n} f(x,y) I_C(x,y)\,\mu_n(dx), \tag{1.7}$$

where f is a symmetric unimodal function on $R^n \times R^m$ and C is a symmetric convex set in R^{n+m}, $y \in R^m$. Note that, for a fixed y, the section of C in the n-space may not be symmetric. The conclusions (1.6) are shown to be valid also if

$$h(y) = \int_{R^n} f_1(x,y) f_2(x,y)\,\mu_n(dx), \tag{1.8}$$

where f_1 and f_2 are symmetric unimodal functions on $R^n \times R^m$. Note now that $f_1(x,y)f_2(x,y)$ may not be unimodal on $R^n \times R^m$. A further generalization is given in Corollary 1. All these results are then extended by replacing the symmetry condition by G^*-invariance for a suitable group G^* of transformations. This is the main result in this paper and it is given in Theorem 1. This generalizes Mudholkar's theorem. The question of replacing μ_n by a more general measure ν is also studied.

A special case of our results shows that a marginal function (i.e., when a subset of the variables are integrated out) of a symmetric unimodal function is symmetric and "ray-unimodal" (i.e., (1.6) holds); however, some examples are given to indicate that a marginal function of a unimodal function need not be unimodal, even when the symmetry condition is assumed.

2. **The main generalization of Anderson's theorem**. Let G_1 and G_2 be groups of measurable one-to-one transformations of $R^n \to$ onto R^n and $R^m \to$ onto R^m, respectively. Let G^* be a subgroup of $G_1 \times G_2$ satisfying the following:

CONDITION A. Given any $g_2 \in G_2$ there exists $g_1 \in G_1$ such that $(g_1, g_2) \in G^*$.

Furthermore, assume the following:

CONDITION B. The group G_1 is Lebesgue measure-preserving.

THEOREM 1. *Let $f_i(x,y)$ $(i = 1, \ldots, k)$ be G^*-invariant unimodal functions on $R^n \times R^m$, $x \in R^n$, $y \in R^m$. Assume that for each $y_1, \ldots, y_k$ in R^m*

$$h(y_1, \ldots, y_k) \equiv \int_{R^n} \prod_{i=1}^{k} f_i(x, y_i)\, \mu_n(dx) < \infty. \tag{2.1}$$

Then

$$h(gy_1, \ldots, gy_k) = h(y_1, \ldots, y_k) \tag{2.2}$$

for any $g \in G_2$, *and*

$$h(y_1^*, \ldots, y_k^*) \geqslant h(y_1, \ldots, y_k), \tag{2.3}$$

where

$$y_i^* = \sum_{j=1}^{\gamma} \lambda_j g_{2j} y_i, \tag{2.4}$$

g_{2j}*'s are in* G_2, γ *is any positive integer, and* $(\lambda_1, \ldots, \lambda_\gamma) \in P_\gamma$, *the* γ*-dimensional probability simplex.*

Proof. For $0 < u_i < \infty$, define

$$D_i(u_i) = \{(x, y): f_i(x, y) \geqslant u_i\}, \tag{2.5}$$

$$D_i(u_i, y) = \{x: (x, y) \in D_i(u_i)\}, \tag{2.6}$$

$i = 1, \ldots, k$. By Fubini's theorem

$$h(y_1, \ldots, y_k) = \int_0^\infty \cdots \int_0^\infty \left[\int_{R^n} \prod_{i=1}^{k} I_{D_i(u_i, y_i)}(x) \mu_n(dx) \right] \prod_{i=1}^{k} du_i \tag{2.7}$$

$$= \int_0^\infty \cdots \int_0^\infty \left[\mu_n \left\{ \bigcap_{i=1}^{k} D_i(u_i, y_i) \right\} \right] du_1, \ldots, du_k. \tag{2.8}$$

Note now

$$\bigcap_{i=1}^{k} D_i(u_i, y^*) \supset \sum_{j=1}^{\gamma} \lambda_j \left[\bigcap_{i=1}^{k} D_i(u_i, g_{2j} y_i) \right]. \tag{2.9}$$

This follows from the fact that the sets $D_i(u_i)$ are convex. Then, from Brunn-Minkowski's inequality, we get

$$\mu_n \left[\bigcap_{i=1}^{k} D_i(u_i, y_i^*) \right] \geqslant \mu_n \left[\sum_{j=1}^{\gamma} \lambda_j \left\{ \bigcap_{i=1}^{k} D_i(u_i, g_{2j} y_i) \right\} \right] \tag{2.10}$$

$$\geqslant \min_{1 < j < \gamma} \left| \mu_n \left\{ \bigcap_{i=1}^{k} D_i(u_i, g_{2j} y_i) \right\} \right|. \tag{2.11}$$

By Condition A there exists $g_{1j}^{-1} \in G_1$ such that $(g_{1j}^{-1}, g_{2j}^{-1}) \in G^*$. Since f_i is G^*-invariant,

$$g_{1j}^{-1} D_i(u_i, g_{2j} y_i) = D_i(u_i, y_i) \tag{2.12}$$

and

$$g_{1j}^{-1} \left[\bigcap_{i=1}^{k} D_i(u_i, g_{2j} y_i) \right] = \bigcap_{i=1}^{k} D_i(u_i, y_i). \tag{2.13}$$

Since G_1 is Lebesgue measure-preserving,

$$\mu_n\left[\bigcap_{i=1}^{k} D_i(u_i, g_{2j}y_i)\right] = \mu_n\left[\bigcap_{i=1}^{k} D_i(u_i, y_i)\right], \tag{2.14}$$

$j = 1, \ldots, \gamma$. Now we get (2.3) from (2.8), (2.11) and (2.14). The result (2.2) follows from (2.8) and (2.13).

COROLLARY 1. *Let* $f_i(x, y)$ $(i = 1, \ldots, k)$ *be symmetric (about the origin) unimodal functions on* $R^n \times R^m$, $x \in R^n, y \in R^m$. *Assume that* (2.1) *holds for each* $y_1, \ldots, y_k$ *in* R^m. *Then*

$$h(y_1, \ldots, y_k) = h(-y_1, \ldots, -y_k), \tag{2.15}$$

and

$$h(\lambda y_1, \ldots, \lambda y_k) \geqslant h(y_1, \ldots, y_k), \tag{2.16}$$

$0 \leqslant \lambda \leqslant 1$.

PROOF. Define G_1 and G_2 to be the groups of sign-change transformations on R^n and R^m, respectively. Define G^* to be the subgroup of $G_1 \times G_2$ consisting of two elements $(+1, +1)$, $(-1, -1)$. Then any y_i^*, defined in (2.4), can be expressed as λy_i, where $|\lambda| \leqslant 1$. With these specializations the desired results follow from Theorem 1.

REMARK 1. Brunn-Minkowski's inequality states that for any two measurable sets A_1 and A_2 in R^n

$$\mu_n(\theta_1 A_1 + \theta_2 A_2) \geqslant \left[\theta_1 \mu_n^{1/n}(A_1) + \theta_2 \mu_n^{1/n}(A_2)\right]^n, \tag{2.17}$$

where $(\theta_1, \theta_2) \in P_2$. We have used this inequality in (2.11). However, instead of using the full strength of this inequality we have used the following property of μ_n:

$$\mu_n(\theta_1 A_1 + \theta_2 A_2) \geqslant \min\left[\mu_n(A_1), \mu_n(A_2)\right]. \tag{2.18}$$

So Theorem 1 will hold if we replace μ_n by a measure ν on R^n such that ν is G_1-invariant and for any two convex sets A_1, A_2 in R^n

$$\nu(\theta_1 A_1 + \theta_2 A_2) \geqslant \min\left[\nu(A_1), \nu(A_2)\right], \tag{2.19}$$

$\theta = (\theta_1, \theta_2) \in P_2$.

REMARK 2. It is seen from Corollary 1 that the unimodality assumption in Anderson's theorem is greatly relaxed. It can be further relaxed by considering the integrand in (2.1) as a function f which is a positive linear combination of finite products of symmetric unimodal functions. The conclusions of Corollary 1 will still hold. This leads essentially to a generalization of Sherman's result (1955).

REMARK 3. Consider a measure G on R^{mk} such that

$$\int h(y_1, \ldots, y_k) G(dy_1, \ldots, dy_k) < \infty. \tag{2.21}$$

Define

$$f(x, \lambda) \equiv \int \prod_{i=1}^{k} f_i(x, \lambda y_i) G(dy_1, \ldots, dy_k). \tag{2.22}$$

Then, under the assumptions in Corollary 1, it follows that

$$\int f(x, \lambda)\, \mu_n(dx) \geqslant \int f(x, 1)\, \mu_n(dx), \tag{2.23}$$

for $0 \leqslant \lambda \leqslant 1$. This leads to a generalization of Theorem 2 of Anderson (1955).

REMARK 4. Let

$$G_1^* \equiv \{ g_1 \in G_1 \colon (g_1, g_2) \in G^* \text{ for some } g_2 \in G_2 \}. \tag{2.24}$$

Then, instead of Condition B, it is sufficient to assume that μ_n is G_1^*-invariant in order to prove Theorem 1.

3. **Some special cases.** In this section we derive some useful special cases of Theorem 1 and study the marginal function of a unimodal function.

THEOREM 2. *Let G be a linear Lebesgue measure-preserving group of one-to-one transformations of R^n onto R^n. Let $p_i(x)$ ($i = 1, \ldots, k$) be G-invariant unimodal functions on R^n. Assume that*

$$h(y_1, \ldots, y_s) \equiv \int \prod_{i=1}^{s} p_i(x + y_i) \prod_{i=s+1}^{k} p_i(x)\, \mu_n(dx) \tag{3.1}$$

for all $y_1, \ldots, y_s$ in R^n, $0 < s \leqslant k$. Then

$$h(y_1, \ldots, y_s) = h(gy_1, \ldots, gy_s) \tag{3.2}$$

for all $g \in G$, and

$$h(y_1^*, \ldots, y_s^*) \geqslant h(y_1, \ldots, y_s), \tag{3.3}$$

where $y_i^* = \sum_{j=1}^{\gamma} \lambda_j g_j y_i$, γ is any positive integer, g_j's are in G, and $(\lambda_1, \ldots, \lambda_\gamma) \in P_\gamma$.

PROOF. The result is obtained easily by specializing Theorem 1 as follows.

$$\begin{aligned} G_1 = G_2 = G, \qquad & G^* = \{(g, g) \colon g \in G\} \subset G \times G, \\ f_i(x, y) = p_i(x + y), \qquad & i = 1, \ldots, s, \\ = p_i(x), \qquad & i = s + 1, \ldots, k, \\ m = n. & \end{aligned} \tag{3.4}$$

REMARK 5. Mudholkar's theorem follows from Theorem 2. To see this, define

$$k = 2, \quad s = 1, \quad p_1(x + y) = f(x + y), \quad p_2(x) = I_E(x). \tag{3.5}$$

REMARK 6. Theorem 2 can be extended using the idea in Remark 2.

COROLLARY 2. *Let $f(x, y)$ be a symmetric unimodal function on $R^n \times R^m$, $x \in R^n$, $y \in R^m$. Let C be a symmetric convex set in R^{n+m}. Assume that*

$$f_1(y) \equiv \int_{R^n} f(x, y) I_C(x, y)\, \mu_n(dx) < \infty \tag{3.6}$$

for all $y \in R^m$. *Then*

$$f_1(y) = f_1(-y), \tag{3.7}$$

and

$$f_1(\lambda y) \geqslant f_1(y), \tag{3.8}$$

for $0 \leqslant \lambda \leqslant 1, y \in R^m$.

PROOF. This follows from Corollary 1, by taking $k = 2$, $f_1(x, y) = f(x, y)$, $f_2(x, y) = I_C(x, y)$.

REMARK 7. Note that f_1, defined in (3.6), is a unimodal function if $m = 1$. However, this result is not true if $m > 1$, as shown by Example 1, which is basically due to Anderson (see Sherman (1955)). In general, f_1, defined in (3.6), need not be unimodal even when $m = 1$ if the symmetry condition is dropped; this is shown in Example 2.

EXAMPLE 1. For $(x, y) \in R^2$, define $f(x, y) = I_A(x) I_B(y) g(x + y)$, where

$$g(t) = \begin{cases} 3, & \text{if } |t_1| \leqslant 1, |t_2| \leqslant 1, \\ 2, & \text{if } |t_1| \leqslant 1, 1 < |t_2| \leqslant 5, \\ 0, & \text{elsewhere,} \end{cases}$$

$t = (t_1, t_2)$, and

$$A = \{x = (x_1, x_2): |x_1| \leqslant 1, |x_2| \leqslant 1\},$$
$$B = \{y = (y_1, y_2): |y_1| \leqslant 2, |y_2| \leqslant 5\}.$$

Then f is a symmetric unimodal function on $R^2 \times R^2$. Define

$$f_1(y) = \int_{R^2} f(x, y)\, dx = I_B(y) \int_A g(x + y)\, dx.$$

Note now $f_1(0.5, 4) = f_1(1, 0) = 6$, but $f_1(0.75, 2) < 6$, and $(0.75, 2) = \frac{1}{2}(0.5, 4) + \frac{1}{2}(1, 0)$. Thus f_1 is not unimodal on R^2.

EXAMPLE 2. For x, y in R^1, define

$$f(x, y) = \begin{cases} 3, & 0 \leqslant x \leqslant y, 0 \leqslant y < 1, \\ 2, & 0 \leqslant x \leqslant y, 1 \leqslant y \leqslant 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$f_1(y) \equiv \int_{-\infty}^{\infty} f(x, y)\, dx = \begin{cases} 3y, & 0 \leqslant y < 1, \\ 2y, & 1 \leqslant y \leqslant 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that f_1 is not unimodal on R^1 although f is unimodal on $R^1 \times R^1$.

REFERENCES

T. W. Anderson (1955), *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. **6**, 170–176. MR **16**, 1005.

G. S. Mudholkar (1966), *The integral of an invariant unimodal function over an invariant convex set–An inequality and applications*, Proc. Amer. Math. Soc. **17**, 1327–1333. MR **34** #7741.

S. Sherman (1955), *A theorem on convex sets with applications*, Ann. Math. Statist. **26**, 763–767. MR **17**, 655.

INSTITUTE FOR MATHEMATICAL STUDIES IN SOCIAL SCIENCES, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305

Current address: Department of Theoretical Statistics, University of Minnesota, Minneapolis, Minnesota 55455