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A SIMPLE PROOF OF THE HOBBY-RICE THEOREM[1]

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ABSTRACT. This paper presents a simple proof of the following theorem due to Hobby and Rice.

THEOREM. *Let* $\{\varphi_i(x)\}_{i=1}^n$ *be n real functions in* $L^1(d\mu;[0,1])$, *where* μ *is a finite, nonatomic, real measure. Then there exist* $\{\xi_i\}_{i=1}^r$, $r \leqslant n$, $0 = \xi_0 < \xi_1 < \cdots < \xi_r < \xi_{r+1} = 1$ *such that*

$$\sum_{j=1}^{r+1} (-1)^j \int_{\xi_{j-1}}^{\xi_j} \varphi_i(x)\, d\mu(x) = 0, \qquad i = 1, \ldots, n.$$

A matrix version of the above theorem is also proven. These results are of importance in the study of L^1-approximation.

1. **Introduction.** The purpose of this short note is to present a simple proof of the Hobby-Rice Theorem, which arises in the study of L^1-approximation.

THEOREM A (HOBBY, RICE [2]). *Let* $\{\varphi_i(x)\}_{i=1}^n$ *be n real functions in* $L^1(d\mu;[0,1])$, *where* μ *is a finite, nonatomic, real measure on* $[0,1]$. *Then there exist* $\{\xi_i\}_{i=1}^r$, $r \leqslant n$, $\xi_0 = 0 < \xi_1 < \cdots < \xi_r < \xi_{r+1} = 1$ *such that*

$$\sum_{j=1}^{r+1} (-1)^j \int_{\xi_{j-1}}^{\xi_j} \varphi_i(x)\, d\mu(x) = 0, \qquad i = 1, \ldots, n.$$

The main difficulty in the proof of Theorem A, as presented in [2], is the construction of an odd continuous mapping through which the result may then be obtained via an application of the Borsuk Antipodality Theorem, or some equivalent version thereof. This construction is herein obtained in an exceedingly simple manner, answering a question posed by Cheney [1].

The author's interest in this result originated in an attempt to prove an analogue of Theorem A for matrices. To state the result, the following definition (also found in [5]) is given.

DEFINITION. Given $0 = j_0 < j_1 < \cdots < j_n < j_{n+1} = M + 1$ and a vector $\mathbf{z} \in \mathbf{R}^M$, we say that $\mathbf{z}$ alternates between $j_1, \ldots, j_n$ provided that there exists a single sign σ, $\sigma = +1$ or -1, such that $z_l = (-1)^i \sigma$, $j_{i-1} < l < j_i$, $i = 1, \ldots, n+1$.

Within this setting, the following result obtains.

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THEOREM B. *Let A be a real $n \times M$ matrix, $n < M$. Then there exists an M-vector $\mathbf{z}$, which alternates between some $\{j_i\}_{i=1}^n$, $1 \leqslant j_1 < \cdots < j_n \leqslant M$, for which* $\|\mathbf{z}\|_\infty = \max_{i=1,\ldots,M}|z_i| = 1$, *and such that* $A\mathbf{z} = \mathbf{0}$.

It should be noted that if $d\mu$ is a nonnegative measure and $\{\varphi_i(x)\}_{i=1}^n$ is a weak Chebyshev system, then Theorem A and generalizations thereof may be found in Krein [4] (see also Karlin and Studden [3, p. 263]).

2. **Proofs.** Our proofs of Theorems A and B utilize the following version of the Borsuk Antipodality Theorem (cf. [6]).

THEOREM C. *Let Ω be a bounded, open, symmetric neighborhood of* 0 *in* $\mathbf{R}^{n+1}$, *and $T \in C(\partial\Omega, \mathbf{R}^n)$, with T odd on $\partial\Omega$. Then there exists $\mathbf{x}^* \in \partial\Omega$ for which* $T(\mathbf{x}^*) = \mathbf{0}$.

PROOF OF THEOREM A. Consider $S^n = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \colon \sum_{i=1}^{n+1}|x_i| = 1\}$, and define $y_0(\mathbf{x}) = 0$, $y_j(\mathbf{x}) = \sum_{k=1}^{j}|x_k|$, $j = 1, \ldots, n+1$. Let $T(\mathbf{x}) = (T_1(\mathbf{x}), \ldots, T_n(\mathbf{x}))$, where

$$T_i(\mathbf{x}) = \sum_{j=1}^{n+1} (\operatorname{sgn} x_j) \int_{y_{j-1}(\mathbf{x})}^{y_j(\mathbf{x})} \varphi_i(x)\, d\mu(x), \qquad i = 1, \ldots, n.$$

Thus $T \in C(S^n, \mathbf{R}^n)$, and T is odd, i.e., $T(-\mathbf{x}) = -T(\mathbf{x})$. An application of Theorem C proves Theorem A.

PROOF OF THEOREM B. Let $S_M^n = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \colon \sum_{i=1}^{n+1}|x_i| = M\}$. For each $\mathbf{x} \in S_M^n$, set $y_0(\mathbf{x}) = 0$, $y_j(\mathbf{x}) = \sum_{k=1}^{j}|x_k|$, $j = 1, \ldots, n+1$. Let $g(\mathbf{x}; t)$ be the unique continuous, piecewise linear function which takes on the value $2\sum_{k=1}^{j} x_k$ at its vertex $y_j(\mathbf{x}) = \sum_{k=1}^{j}|x_k|$, $j = 1, \ldots, n+1$, and such that $g(\mathbf{x}; 0) = 0$. (The discontinuities of $\partial g(\mathbf{x}; t)/\partial t$ occur only at the $y_j(\mathbf{x})$, $j = 1, \ldots, n$.) In other words,

$$g(\mathbf{x}; t) = 2\sum_{i=1}^{n+1}\left[(\operatorname{sgn} x_i)\int_{y_{i-1}(\mathbf{x})}^{y_i(\mathbf{x})} (t-s)_+^0\, ds\right].$$

Define

$$f_{\mathbf{x}}(t) = \min\{\max[g(\mathbf{x}; t), -1], 1\} = \max\{\min[g(\mathbf{x}; t), 1], -1\},$$

and note that

(1) $|f_{\mathbf{x}}(t)| \leqslant 1$, $t \in [0, M]$,

(2) $f_{\mathbf{x}}(t)$ is a continuous piecewise linear function on $[0, M]$,

(3) $f_{-\mathbf{x}}(t) = -f_{\mathbf{x}}(t)$,

(4) for each $t \in [0, M]$, $f_{\mathbf{x}}(t)$ is a continuous function of $\mathbf{x} \in S_M^n$.

Now, set $z_i(\mathbf{x}) = f_{\mathbf{x}}(i)$, $i = 1, \ldots, M$, and $\mathbf{z}(\mathbf{x}) = (z_1(\mathbf{x}), \ldots, z_M(\mathbf{x}))$ for each $\mathbf{x} \in S_M^n$. $A\mathbf{z}(\mathbf{x})$ is a continuous odd map of S_M^n into $\mathbf{R}^n$. From Theorem C, there exists an $\mathbf{x}^* \in S_M^n$ for which $A\mathbf{z}(\mathbf{x}^*) = \mathbf{0}$. By construction $\mathbf{z}(\mathbf{x}^*) = \mathbf{z}^*$ alternates at at most n components, and $\|\mathbf{z}^*\|_\infty = 1$. We may regard $\mathbf{z}^*$ as alternating at exactly n components by a judicious choice of the components.

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