PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

NECESSARY AND SUFFICIENT CONDITIONS FOR L^1 CONVERGENCE OF TRIGONOMETRIC SERIES

JOHN W. GARRETT AND ČASLAV V. STANOJEVIĆ

ABSTRACT. It is shown that for the class of cosine series satisfying $a(n)\log n = o(1)$ and $\Delta a(n) > 0$ that integrability and L^1 convergence occur together. Relaxing the monotonicity to bounded variation we show that our previous result cannot be extended.

It is well known that the condition $a(n)\log n = o(1)$ is both necessary and sufficient for L^1 convergence for some classes of Fourier cosine series. Here we show, for the class of cosine series satisfying $a(n)\log n = o(1)$ and $\Delta a(n) \geqslant 0$, that integrability and L^1 convergence occur together. Relaxing the monotonicity to bounded variation we show that our previous result [1] cannot be extended. Finally we show that a cosine series with $\Delta a_n \geqslant 0$ is integrable if the norm of the derivative of the partial sums of its conjugate series are bounded.

In what follows $f(x) = \lim_{n\to\infty} S_n(x)$ where

$$S_n(x) = \frac{1}{2}a(0) + \sum_{k=1}^{n}\left[a(k)\cos kx + b(k)\sin kx\right].$$

We denote $\sigma_n(x) = 1/(n+1)\sum_{k=0}^{n} S_k(x)$, and $\overline{S_n'}(x)$ is the derivative of the conjugate of $S_n(x)$.

THEOREM 1. *Let $a(n)\log n = o(1)$, $b(n)\log n = o(1)$, $\Delta a(n) \geqslant 0$, and $\Delta b(n) \geqslant 0$. Then $\|\overline{S_n'}\| = o(n)$.*

PROOF.

$$\begin{aligned}\|\overline{S_n'}\| &= \left\|\sum_{k=1}^{n}\left[ka(k)\cos kx + kb(k)\sin kx\right]\right\| \\ &= \Bigg\|\sum_{k=1}^{n-1}\left\{\left[k\Delta a(k) - a(k+1)\right]\left[D_k(x) - \frac{1}{2}\right]\right. \\ &\qquad\qquad \left.+\left[k\Delta b(k) - b(k+1)\right]\overline{D_k}(x)\right\} \\ &\qquad\qquad + na(n)\left[D_n(x) - \frac{1}{2}\right] + nb(n)\overline{D_n}(x)\Bigg\|\end{aligned}$$

Received by the editors August 6, 1975.
AMS (MOS) subject classifications (1970). Primary 42A20, 42A32.
Key words and phrases. L^1 convergence of Fourier series, Fejér sums.

$$\leqslant B \sum_{k=1}^{n-1} k\Delta a(k)\log k + B \sum_{k=1}^{n-1} a(k+1)\log k$$
$$+ B \sum_{k=1}^{n-1} k\Delta b(k)\log k + B \sum_{k=1}^{n-1} b(k+1)\log k$$
$$+ Bna(n)\log n + Bnb(n)\log n$$

where $D_n(x)$ and $\overline{D_n}(x)$ are the Dirichlet and conjugate Dirichlet kernels, and B is an absolute constant arising from the fact that

$$\|D_n(x) - 1/2\| = O(\log n) \quad \text{and} \quad \|\overline{D_n}(x)\| = O(\log n).$$

Four terms are $o(n)$ since $a(n)\log n = o(1)$, $b(n)\log n = o(1)$, and the $(C,1)$ method is regular. Thus,

$$\|\overline{S_n'}\| \leqslant B \sum_{k=1}^{n-1} k[\Delta a(k) + \Delta b(k)]\log k + o(n)$$
$$= B \sum_{k=1}^{n-1} \{k\Delta([a(k) + b(k)]\log k)$$
$$+ k[a(k+1) + b(k+1)]\log[(k+1)/k]\} + o(n)$$
$$= B \sum_{k=1}^{n-1} [a(k) + b(k)]\log k - B(n-1)[a(n) + b(n)]\log n$$
$$+ B \sum_{k=1}^{n-1} [a(k+1) + b(k+1)]\log(1 + 1/k)^k + o(n)$$
$$= o(n)$$

since

$$[a(n) + b(n)]\log n = o(1),$$

the $(C,1)$ method is regular, and $\log(1 + 1/k)^k$ converges to one.

Corollary 1. *Let $a(n)\log n = o(1)$, $b(n)\log n = o(1)$, $\Delta a(n) \geqslant 0$, and $\Delta b(n) \geqslant 0$. Then f is integrable if and only if S_n converges to f in L^1 metric.*

Proof. "If": Obvious. "Only if": It is well known that if f is integrable then σ_n converges to f in L^1 metric. Hence $\|S_n - f\| \leqslant \|S_n - \sigma_n\| + \|\sigma_n - f\|$. But $\|S_n - \sigma_n\| = 1/(n+1)\|\overline{S_n'}\| = o(1)$.

The following propositions are now apparent.

Proposition 1. *Let f be integrable. Then S_n converges to f in L^1 metric if and only if $\|\overline{S_n'}\| = o(n)$.*

Proposition 2. *Let $\|\overline{S_n'}\| = o(n)$. Then f is integrable if and only if S_n converges to f in L^1 metric.*

Indeed, for any sequence, $A(n)$, the following proposition holds.

PROPOSITION 3. *Let $A(n)$ be a sequence of positive numbers.*

(1) *Let $\|\sigma_n - f\| = o(A(n))$. Then $\|S_n - f\| = o(A(n))$ if and only if $\|\overline{S_n'}\| = o(nA(n))$.*

(2) *Let $\|S_n - f\| = o(A(n))$. Then $\|\sigma_n - f\| = o(A(n))$ if and only if $\|\overline{S_n'}\| = o(nA(n))$.*

(3) *Let $\|\overline{S_n'}\| = o(nA(n))$. Then $\|\sigma_n - f\| = o(A(n))$ if and only if $\|S_n - f\| = o(A(n))$.*

It is clear that Proposition 3 contains Proposition 1 as the special case where $A(n) = 1$. Also, since $\|\overline{S_n'}\| = o(1)$ is equivalent to f being constant, we have the following special case. Let $n\|S_n - f\| = o(1)$ $[n\|\sigma_n - f\| = o(1)]$. Then $n\|\sigma_n - f\| = o(1)$ $[n\|S_n - f\| = o(1)]$ if and only if f is constant.

In Corollary 1 we required $\Delta a(n) \geqslant 0$. Several results on L^1 convergence of cosine series are known that only require bounded variation of $a(n)$, that is, $\sum_{n=1}^{\infty} |\Delta a(n)| < \infty$. It is well known that if $a(n) = o(1)$ and $a(n)$ is quasi-convex $(\sum_{n=1}^{\infty}(n+1)|\Delta^2 a(n)| < \infty)$ that S_n converges to f in L^1 metric if and only if $a(n)\log n = o(1)$. Using an inequality of Sidon, Telyakovskii [2] has proved the following theorem where quasi-convexity is relaxed.

THEOREM A. *Let $f(x) = \lim_{n\to\infty} S_n(x)$ where $b(n) = 0$ and $a(n) = o(1)$. Let numbers $A(n)$ exist such that $\Delta A(n) \geqslant 0$, $\sum_{n=0}^{\infty} A(n) < \infty$, and $|\Delta a(n)| \leqslant A(n)$ for all n. Then S_n converges to f in L^1 metric if and only if $a(n)\log n = o(1)$.*

Recently we [1] found a condition necessary and sufficient for a modification of S_n to converge to f in L^1 metric.

THEOREM B. *Let*

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a(j) \cos kx,$$

$b(n) = 0$, $a(n) = o(1)$, and $\sum_{n=1}^{\infty} |\Delta a(n)| < \infty$. Then g_n converges to f in L^1 metric if and only if

for $\varepsilon > 0$ there exists $\delta > 0$ (independent of n) such that

$$\text{(C)} \qquad \int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| < \varepsilon.$$

As a corollary we extended Telyakovskii's result.

COROLLARY B. *Let $b(n) = 0$, $a(n) = o(1)$, $\sum_{n=1}^{\infty}|\Delta a(n)| < \infty$, and (C) be satisfied. Then S_n converges to f in L^1 metric if and only if $a(n)\log n = o(1)$.*

Here we show that if we require the conditions $a(n) = o(1)$ and $\sum_{n=1}^{\infty}|\Delta a(n)| < \infty$ then Theorem A cannot be extended beyond Corollary B.

THEOREM 2. *Let $b(n) = 0$, $a(n) = o(1)$, $\sum_{n=1}^{\infty}|\Delta a(n)| < \infty$, and $a(n)\log n = o(1)$. Then S_n converges to f in L^1 metric if and only if condition (C) is satisfied.*

PROOF. Using g_n as defined in Theorem B,

$$
\begin{aligned}
\|S_n(x) - f(x)\| &= \left\| \frac{1}{2} a(0) + \sum_{k=1}^{n} a(k)\cos kx - f(x) \right\| \\
&= \left\| \frac{1}{2} a(0) - \frac{1}{2} a(n+1) + \sum_{k=1}^{n} [a(k) - a(n+1)]\cos kx - f(x) \right. \\
&\qquad \left. + \frac{1}{2} a(n+1) + \sum_{k=1}^{n} a(n+1)\cos kx \right\| \\
&= \left\| \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a(j)\cos kx - f(x) + a(n+1)D_n(x) \right\| \\
&= \| g_n(x) - f(x) + a(n+1)D_n(x) \|.
\end{aligned}
$$

But $\|a(n+1)D_n(x)\| = o(1)$, since $a(n)\log n = o(1)$ and $\|D_n(x)\| = O(\log n)$. Thus, S_n converges to f in L^1 metric if and only if g_n converges to f in L^1 metric. We see that the coefficients $a(n)$ satisfy the requirements of Theorem B, so the result follows.

At this point we see that if $|\sum_{n=1}^{\infty} b(n)| < \infty$ then $\|\overline{S_n}\| = O(\|\overline{S_n'}\|)$. For

$$
\begin{aligned}
\|\overline{S_n}\| &= \int_{-\pi}^{\pi} \left| \int_0^x \overline{S_n'}(t)\,dt + \sum_{n=1}^{\infty} b(n) \right| dx \\
&\leqslant \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\overline{S_n'}(t)|\,dt\,dx + 2\pi \left| \sum_{n=1}^{\infty} b(n) \right| \\
&= 2\pi \|\overline{S_n'}\| + 2\pi \left| \sum_{n=1}^{\infty} b(n) \right|.
\end{aligned}
$$

This leads to integrability conditions for f and $\tilde{f}$, the conjugate of f.

PROPOSITION 4. *Let* $|\sum_{n=1}^{\infty} b(n)| < \infty$. *If* $\|\overline{S_n'}\| = O(1)$ *then* $\tilde{f} \in L^1$. *If in addition we require* $\Delta a(n) \geqslant 0$, $\Delta b(n) \geqslant 0$, *then* $f \in L^1$.

REFERENCES

1. J. W. Garrett and Č. V. Stanojević, *On L^1 convergence of certain cosine sums*, Proc. Amer. Math. Soc. **54** (1976), 101–105.

2. S. A. Teljakovskiĭ, *Concerning a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki **14** (1973), 317–328 = Math. Notes **14** (1973), 742–748 (1974). MR **48** #6798.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW ORLEANS, NEW ORLEANS, LOUISIANA 70122

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, ROLLA, MISSOURI 65401