PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

MONOTONE AND OPEN MAPPINGS ONTO ANR'S

JOHN J. WALSH

ABSTRACT. Let M be either a compact, connected p.l. manifold of dimension at least three or a compact, connected Hilbert cube manifold and let Y be a compact, connected ANR (= absolute neighborhood retract). The main results of this paper are: (i) a mapping f from M to Y is homotopic to a monotone mapping from M onto Y if and only if $f_*: \pi_1(M) \to \pi_1(Y)$ is surjective; (ii) a mapping f from M to Y is homotopic to an open mapping from M onto Y if and only if $f_*(\pi_1(M))$ has finite index in $\pi_1(Y)$.

In [5] and [6], the author showed that a mapping f from a p.l. manifold M^m $(m \geqslant 3)$ to a polyhedron P is homotopic to a monotone mapping from M onto P if $f_*: \pi_1(M) \to \pi_1(P)$ is surjective and is homotopic to an open mapping from M onto P if $f_*(\pi_1(M))$ has finite index in $\pi_1(P)$. Using these results, a recent result on open mappings, and results from infinite dimensional topology, we show that the above results are true if P is only assumed to be an ANR.

Terminology. We will use M^m to denote either an m-dimensional compact, connected, p.l. (= piecewise linear) manifold with or without boundary or, if $m = Q$, a compact, connected Q-manifold (where Q is the Hilbert cube). By a *mapping* we mean a continuous function; a mapping is *open* if the image of each open set is open; a mapping is *monotone* (= UV^0) if each point-inverse is connected. We refer the reader to [1] for the definition of an *absolute neighborhood retract* (= ANR). Spaces are assumed to be separable and metric.

Main results.

THEOREM 1. *Let M be a compact, connected, p.l. manifold with dimension at least three or a compact, connected Q-manifold and let Y be a compact, connected ANR. A mapping f from M to Y is homotopic to a monotone mapping from M onto Y if and only if $f_*: \pi_1(M) \to \pi_1(Y)$ is surjective.*

THEOREM 2. *Let M and Y be as above. A mapping f from M to Y is homotopic to an open mapping from M onto Y if and only if $f_*(\pi_1(M))$ has finite index in $\pi_1(Y)$.*

REMARK. The "only if" half of each of these theorems is well known; see

Received by the editors March 29, 1976.
AMS (MOS) subject classifications (1970). Primary 54C10; Secondary 57C99.
Key words and phrases. Open mapping, monotone mapping, manifold, ANR.

Smale [3] and [4]. It follows from Fact 2 below that if $f_*: \pi_1(M) \to \pi_1(Y)$ is surjective, then f is homotopic to a mapping from M onto Y which is both monotone and open.

We present below several results which will be used in the proofs of the above theorems.

FACT 1. Theorems 1 and 2 are true if Y is assumed to be a polyhedron or a Q-manifold. These cases are "essentially" contained in [5, Theorem 2.0] and [6, Theorem 4.0]. The theorems in [5] and [6] are stated for finite dimensional p.l. manifolds and polyhedra; however, using Chapman's result in [2] that every compact Q-manifold is homeomorphic to the product of a polyhedron and Q, the proofs in [5] and [6] "work" without difficulty.

FACT 2. The main result in [7] is that a monotone mapping f from a compact manifold M^m, $m \geqslant 3$, onto any space Y can be homotoped (by an arbitrarily small homotopy) to a montone open mapping g from M onto Y (with $g^{-1}(y)$ and $f^{-1}(y)$ having the same shape for each $y \in Y$). The proof in [7] "works" equally well if M is a compact Q-manifold.

FACT 3. A "key" step in the proofs depends on West's recent result in [8] that every compact ANR is the CE (= cell-like) image of a Q-manifold.

PROOF OF THEOREM 1. Let g: $W \to Y$ be a CE mapping of a Q-manifold W onto Y and let $\tilde{f}$: $M \to W$ be a "lift" of f with $g \circ \tilde{f}$ homotopic to f (for example, let $\tilde{f} = i \circ f$ where i is a "homotopy inverse" of g). Since g is a homotopy equivalence, we have that $\tilde{f}_*: \pi_1(M) \to \pi_1(W)$ is surjective. Fact 1 implies that $\tilde{f}$ is homotopic to a monotone mapping $\tilde{h}$ from M onto W; letting $h = g \circ \tilde{h}$, h is a monotone mapping of M onto Y homotopic to f.

PROOF OF THEOREM 2. Let $x_0 \in M$, $y_0 \in Y$ with $f(x_0) = y_0$ and let p: $(\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ be the covering projection with

$$p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) = f_*(\pi_1(M, x_0)).$$

Since $f_*(\pi_1(M, x_0))$ has finite index in $\pi_1(Y, y_0)$, we have that $p^{-1}(y_0)$ is finite and, hence, $\tilde{Y}$ is compact; also, $\tilde{Y}$ is an ANR (see [1, Chapter 4, §10]). Let $\tilde{f}$: $(M, x_0) \to (\tilde{Y}, \tilde{y}_0)$ be a lifting of f; it follows that $\tilde{f}_*: \pi_1(M, x_0) \to \pi_1(\tilde{Y}, \tilde{y}_0)$ is onto. Applying Theorem 1 and Fact 2 to $\tilde{f}$, we have that $\tilde{f}$ is homotopic to a monotone open mapping $\tilde{h}$ from M onto $\tilde{Y}$; letting $g = p \circ \tilde{h}$, g is an open mapping from M onto Y homotopic to f.

REFERENCES

1. K. Borsuk, *Theory of retracts*, Monografie Mat., tom 44, PWN, Warsaw, 1967. MR **35** #7306.

2. T. Chapman, *Compact Hilbert cube manifolds and the invariance of Whitehead torsion*, Bull. Amer. Math. Soc. **79** (1973), 52–56. MR **49** #11518.

3. S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. **8** (1957), 604–610. MR **19**, 302.

4. ———, *A note on open maps*, Proc. Amer. Math. Soc. **8** (1957), 391–393. MR **19**, 158.

5. J. Walsh, *Monotone and open mappings on manifolds*. I, Trans. Amer Math. Soc. **209** (1975), 419–432.

6. ———, *Light open and open mappings on manifolds*. II, Trans. Amer. Math. Soc. **217** (1976), 271–284.

7. ———, *Isotoping mappings to open mappings*, Trans. Amer. Math. Soc. (to appear).

8. J. West, *Compact ANR's have finite type* (preprint).

9. D. C. Wilson, *Open mappings on manifolds and a counterexample to the Whyburn conjecture*, Duke Math. J. **40** (1973), 705–716. MR **47** #9522.

10. ———, *On constructing monotone and UV^1 mappings of arbitrary degree*, Duke Math. J. **41** (1974), 103–109. MR **49** #4011.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73069