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S-CLOSED SPACES

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ABSTRACT. A topological space X is said to be S-closed if and only if for every semiopen cover of X there exists a finite subfamily such that the union of their closures cover X. For a compact Hausdorff space, the concept of S-closed is shown to be equivalent to the concepts of extremally disconnected and projectiveness.

A Hausdorff space X is H-closed if and only if for every open cover $\{U_a | a \in \Lambda\}$ there exists a finite subfamily $\{U_{a_i} | i = 1, 2, \ldots, n\}$ such that the union of their closures cover X. In this paper, we expand this concept using semiopen sets.

DEFINITION 1. A set A in a topological space X is semiopen if and only if there exists an open set V such that $V \subset A \subset \overline{V}$, where $\overline{V}$ is the closure of V.

DEFINITION 2. A filterbase $F = \{A_a\}$ *s-converges* to a point $x_0 \in X$ if for each semiopen set V containing x_0 there exists an $A_a \in F$ such that $A_a \subset \overline{V}$.

DEFINITION 3. A filterbase $F = \{A_a\}$ *s-accumulates* to a point $x_0 \in X$ if for each semiopen set V containing x_0 and $A_a \in F$, $A_a \cap \overline{V} \neq \varnothing$.

The corresponding definitions using nets are apparent and will not be stated. An easy consequence of these definitions is

THEOREM 1. *Let F be a maximal filterbase in X. Then F s-accumulates to a point $x_0 \in X$ if and only if F s-converges to x_0.*

DEFINITION 4. A topological space X is S-closed if and only if for every semiopen cover $\{U_a | a \in \Gamma\}$ of X there exists a finite subfamily $\{U_{a_i} | i = 1, 2, \ldots, n\}$ such that the union of their closures cover X.

It is apparent from the definition above that a Hausdorff S-closed space is H-closed. The reader can readily find examples to show that the converse need not be true. Our first result lies in Theorem 2 which characterizes S-closed spaces.

THEOREM 2. *For a topological space the following are equivalent:*

(i) *X is S-closed.*

(ii) *For each family of semiclosed sets $\{F_a\}$ (i.e., each F_a is the complement of a semiopen set) such that $\bigcap (F_a) = \varnothing$, there exists a finite subfamily $\{F_{a_i}\}_{i=1}^n$ such that $\bigcap_{i=1}^n (F_{a_i})^0 = \varnothing$.*

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(iii) *Each filterbase* $F = \{A_a\}$ *s-accumulates to some point* $x_0 \in X$.
(iv) *Each maximum filterbase* F *s-converges.*

PROOF. (i) $\Rightarrow$ (iv). Let $F = \{A_a\}$ be a maximum filterbase. Suppose that F does not s-converge to any point; therefore, by Theorem 1, F does not s-accumulate to any point. This implies that for every $x \in X$, there exists a semiopen set $V(x)$ containing x and an $A_{a(x)} \in F$ such that $A_{a(x)} \cap \overline{V(x)} = \varnothing$. Obviously $\{V(x)|x \in X\}$ is a semiopen cover for X and by hypothesis there exists a finite subfamily such that $\bigcap_{i=1}^{n} \overline{V(x_i)} = X$. Since F is a filterbase, there exists an $A_0 \in F$ such that $A_0 \subset \bigcap_{i=1}^{n} A_{a(x_i)}$. Hence, $A_0 \cap \overline{V(x_i)} = \varnothing$, $1 \leqslant i \leqslant n$, which implies $A_0 \cap (\bigcup_{i=1}^{n} \overline{V(x_i)}) = A_0 \cap X = \varnothing$, contradicting the essential fact that $A_0 \neq \varnothing$.

(iv) $\Rightarrow$ (iii). Each filterbase is contained in a maximal filterbase.

(iii) $\Rightarrow$ (ii). Let $\{F_a\}$ be a collection of semiclosed sets such that $\bigcap F_a = \varnothing$. Suppose that for every finite subfamily, $\bigcap_{i=1}^{n}(F_{a_i})^0 \neq \varnothing$. Therefore $F = \{\bigcap_{i=1}^{n}(F_{a_i})^0 | n \in Z^+, F_{a_i} \in \{F_a\}\}$ forms a filterbase. By hypothesis, F s-accumulates to some point $x_0 \in X$. This implies that for every semiopen set $V(x_0)$ containing x_0, $F_a^0 \cap \overline{V(x_0)} \neq \varnothing$, for every $a \in \Lambda$. Since $x_0 \not\in \bigcap F_a$ there exists an $a_0 \in \Lambda$ such that $x_0 \not\in F_{a_0}$. Hence, x_0 is contained in the semiopen set $X - F_{a_0}$. Therefore,

$$(F_{a_0})^0 \cap \overline{(X - F_{a_0})} = (F_{a_0})^0 \cap \left(X - (F_{a_0})^0\right) = \varnothing,$$

contradicting the fact that F s-accumulates to x_0.

(ii) $\Rightarrow$ (i). Let $\{V_a\}$ be a semiopen covering of X. Then $\bigcap(X - V_a) = \varnothing$. By hypothesis, there exists a finite subfamily such that $\bigcap_{i=1}^{n}(X - V_{a_i})^0 = \bigcap_{i=1}^{n}(X - \overline{V}_{a_i}) = \varnothing$. Therefore, $\bigcap_{i=1}^{n} \overline{V}_{a_i} = X$, and consequently X is S-closed. $\square$

THEOREM 3. *Each S-closed, first countable, regular space is finite.*

PROOF. Let X be an S-closed, first countable, regular space. Suppose, if possible, that X is infinite. Since X is compact, it is not discrete. Thus X has an accumulation point x. Let $\{U_n|n \in N\}$ be a local base at x such that $U_1 = X$, U_n is open in X and $\overline{U_{n+1}} \subset U_n$ for each $n \in N$. Let $\{N_k|k \in N\}$ be a family of pairwise disjoint infinite subsets of N (N = the set of positive integers) such that $\bigcup\{N_k|k \in N\} = N$. For each $k \in N$ we set $V_k = \{x\} \cup \bigcup\{\overline{U}_n - \overline{U}_{n+1}|n \in N_k\}$. Then $\{V_k|k \in N\}$ is a semiopen cover of X. If $n \in N$, then $\bigcup\{\overline{V}_k|k \leqslant n\} \neq X$. Thus X is not S-closed and this is a contradiction. $\square$

COROLLARY. *Each S-closed metrizable space is finite.*

THEOREM 4. *No regular space containing a P-point is S-closed.*

PROOF. The proof is omitted, because it is similar to the proof of Theorem

3, but the index set of cardinality $\aleph_1$ is decomposed into $\aleph_1$ sets each of cardinality $\aleph_1$. $\square$

COROLLARY. *$\beta N - N$ is not S-closed.*

PROOF. Assuming the continuum hypothesis, there are P-points in $\beta N - N$ (W. Rudin). $\square$

Since every compact, countable, regular space is metrizable, we have from the corollary to Theorem 3 the following.

COROLLARY. *Each infinite, S-closed, regular space is uncountable.*

LEMMA. *If Y is a regularly closed subset in an S-closed space X, then Y is S-closed.*

PROOF. The proof is easy and is thus omitted.

THEOREM 5. *Each extremally disconnected, compact space is S-closed.*

PROOF. If X is extremally disconnected, then the closure of an open set is an open set. The interior of a semiopen set is dense in it. We consider $\{\overline{U_t^0} | t \in T\}$ instead of given semiopen cover.

COROLLARY. *βN is S-closed.*

THEOREM 6. *If X is a S-closed regular space, then X is extremally disconnected.*

PROOF. Suppose that X is not extremally disconnected. Then there exists a regular open set $O \subset X$ such that $\overline{O} - O$ and $X - \overline{O}$ are nonempty. Let $x \in \overline{O} - O$. Then for every neighborhood V of x, $V \cap O \neq \varnothing$. Therefore, $F = \{(V \cap O)\}$ forms a filterbase in $\overline{O}$. Since $\overline{O}$ is S-closed, F s-accumulates to some point x_0 in $\overline{O}$. Quite obviously, the filterbase F also converges to x in the usual sense. We claim that $x_0 \not\in \overline{O} - O$; for if it were, then $x_0 \in X - O$ and every member of F would have to intersect $X - O$, an impossibility. Thus, $x_0 \in 0$. There now exists an open set U such that $x_0 \in U \subset \overline{U} \subset O$ and $x \in X - \overline{U}$. But since F converges to x, there must exist a neighborhood of x, say V, such that $(V \cap O) \subset X - \overline{U}$. This then would imply that $(V \cap O) \cap \overline{U} = \varnothing$, contradicting the fact that F s-accumulates to x_0. Therefore, our assumption that X is not extremally disconnected is false, and the theorem follows.

THEOREM 7. *If X is a Hausdorff S-closed space, then X is extremally disconnected.*

PROOF. Although the proof is not identical to that of Theorem 6, it is quite similar and is thus omitted.

COROLLARY. *Let X be a regular compact space. Then X is S-closed if and only if X is extremally disconnected.*

COROLLARY. *The one-point compactifications of discrete spaces are not S-closed spaces.*

COROLLARY. *Each S-closed, scattered, regular space is finite.*

It is well known for a completely regular space X that X is extremally disconnected if and only if βX is extremally disconnected [3, p. 96]. Therefore, we have the following corollaries:

COROLLARY. *For a completely regular space X the following are equivalent:*
(i) *X is extremally disconnected.*
(ii) *βX is extremally disconnected.*
(iii) *βX is S-closed.*

COROLLARY. *For a compact Hausdorff space, the following are equivalent:*
(i) *X is extremally disconnected.*
(ii) *X is projective.*
(iii) *X is S-closed.*

PROOF. The equivalence of (i) and (ii) are well known.

This last useful corollary allows us to see immediately that βN is a S-closed space, whereas βQ and βR are not. Additionally, we see that $\beta Q - Q$ is not S-closed since it is not compact, and $\beta R - R$ is not S-closed because it is not extremally disconnected.

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