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ON H-CLOSED AND MINIMAL HAUSDORFF SPACES

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ABSTRACT. In this article, characterizations of H-closed and minimal Hausdorff spaces are given along with some relating properties.

1. **Introduction.** Letting $\mathbb{S}$ denote a class of topological spaces containing as a subclass the Hausdorff completely normal and fully normal spaces, Professors L. L. Herrington and P. E. Long, in a recent paper [2], gave the following characterization of H-closed spaces: A Hausdorff space Y is H-closed if and only if for every space X in class $\mathbb{S}$, each $g: X \to Y$ with a strongly-closed graph is weakly-continuous. In §3 of this paper we improve upon the sufficiency of this theorem by establishing that a Hausdorff space Y is H-closed if for every space X in class $\mathbb{S}$, each bijection $g: X \to Y$ with a strongly-closed graph is weakly-continuous.

Also, for a set X and function $g: X \to X$, we let $F(g)$ denote the set of fixed points of g (i.e. $F(g) = \{x \in X: x = g(x)\}$) and prove the following of our main theorems in §3.

($*$) A Hausdorff space (X, τ) is H-closed if and only if for each topology τ^* on X with (X, τ^*) in class $\mathbb{S}$ for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed in X for each bijection $g: (X, \tau^*) \to (X, \tau)$ with a strongly-closed graph.

($**$) A Hausdorff space (X, τ) is H-closed if and only if for each topology τ^* on X with (X, τ^*) in class $\mathbb{S}$ for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g) = X$ whenever $F(g)$ is dense in X and $g: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph.

In [3], Professors Herrington and Long have proved the following theorem: Let $g: X \to Y$ be a function and let Y be minimal Hausdorff. If g has a strongly-closed graph, then g is continuous.

In §4 of this paper, we prove as another of our main results the following strong sufficiency to their theorem.

($***$) A Hausdorff space Y is minimal Hausdorff if for every space X in class $\mathbb{S}$, each bijection $g: X \to Y$ with a strongly-closed graph is continuous.

In §5, we offer some examples.

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2. **Preliminaries.** We denote by cl $[K]$ the *closure* of a subset K of a topological space.

2.1. DEFINITION [6]. A point x is in the *θ-closure of a subset* K of a space if each open subset V of the space with $x \in V$ satisfies $K \cap \text{cl}\,[V] \neq \varnothing$. In this case we write $x \in \theta\text{-cl}[K]$.

2.2. DEFINITION [6]. A point x in a space is in the *θ-adherence* of a filterbase $\mathfrak{W}$ on the space if $x \in \theta\text{-cl}[F]$ for each $F \in \mathfrak{W}$. In this case we will sometimes say that the filterbase $\mathfrak{W}$ *θ-adheres to* x and use the notation $x \in \theta\text{-adh}\,\mathfrak{W}$.

2.3. DEFINITION [4]. A function $g: X \to Y$ is weakly continuous if for each $x \in X$ and each W open in Y about $g(x)$, there exists a V open in X about x with $g(V) \subset \text{cl}\,[W]$.

We prove the following theorem which we use later in the paper.

2.1. THEOREM. *A function $g: X \to Y$ is weakly-continuous if and only if $g(\text{cl}\,[K]) \subset \theta\text{-cl}[g(K)]$ for each $K \subset X$.*

PROOF. *Necessity.* Let $y \in g(\text{cl}\,[K])$ where $K \subset X$ and $g: X \to Y$ is weakly-continuous. Let $x \in \text{cl}\,[K]$ with $g(x) = y$ and let W be open about y. There is a V open about x satisfying $g(V) \subset \text{cl}\,[W]$. So

$$\varnothing \neq g(V \cap K) \subset g(V) \cap g(K) \subset \text{cl}\,[W] \cap g(K)$$

and the necessity is proved.

Sufficiency. Suppose $g: X \to Y$ satisfies the inclusion of the theorem, let $x \in X$ and let W be open in Y about $g(x)$. Then $W \cap \theta\text{-cl}[g(X) - \text{cl}\,[W]] = \varnothing$. Consequently, $g(x) \notin \theta\text{-cl}[g(X - g^{-1}(\text{cl}\,[W]))]$. Thus

$$g(x) \notin g(\text{cl}[X - g^{-1}(\text{cl}[W])]) \quad \text{and} \quad x \notin \text{cl}[X - g^{-1}(\text{cl}\,[W])].$$

This implies that there is a V open about x satisfying $V \subset g^{-1}(\text{cl}\,[W])$ and the proof is complete.

2.4. DEFINITION [2]. A function $g: X \to Y$ has a *strongly-closed graph* if for each $(x,y) \notin G(g)$, the graph of g, there exist open sets $V \subset X$ and $W \subset Y$ containing x and y, respectively, such that $(V \times \text{cl}\,[W]) \cap G(g) = \varnothing$.

We give without proof the following theorem which we use in the sequel.

2.2. THEOREM. *A function $g: X \to Y$ has a strongly-closed graph if and only if $\{g(x)\} = \bigcap_\Sigma \theta\text{-cl}[g(V)]$ for each $x \in X$ and each (some) open set base Σ at x.*

2.5. DEFINITION. If x_0 is a point in a space X and $\mathfrak{W}$ is a filterbase on X, then $\{A \subset X: x_0 \in X - A$ or $F \cup \{x_0\} \subset A$ for some $F \in \mathfrak{W}\}$ is a topology on X which will be called *the topology on X associated with x_0* and $\mathfrak{W}$. X equipped with this topology will be called *the space associated with x_0 and $\mathfrak{W}$.*

The space associated with a filterbase on a space and a point x_0 in the space will be used frequently in this paper. The following result is easily proved.

2.3. THEOREM. *Let X be a space, let $x_0 \in X$ and let $\mathfrak{W}$ be a filterbase on X which has an empty intersection on $X - \{x_0\}$. The space X associated with x_0 and $\mathfrak{W}$ is in class $\mathbb{S}$.*

3. ***H*-closed spaces.** We use the following characterization of H-closed spaces.

3.1. DEFINITION [6]. A Hausdorff space is *H-closed* if each filterbase on the space θ-adheres to some point in the space.

The sufficiency of our next theorem improves upon the sufficiency of the main result in [2]. We also give a different proof of the necessity of that main result based on the characterization of weakly-continuous functions in Theorem 2.1 above.

3.1. THEOREM. *A Hausdorff space Y is H-closed if and only if for every space X in class $\mathbb{S}$, each bijection $g: X \to Y$ with a strongly-closed graph is weakly-continuous.*

PROOF. *Strong necessity* [2]. Let X be any space, let Y be H-closed, let $g: X \to Y$ have a strongly-closed graph and let $K \subset X$. For $y \in g(\mathrm{cl}\,[K])$, choose $x \in \mathrm{cl}\,[K]$ with $g(x) = y$ and let Σ be an open set base at x. Then $\mathcal{W} = \{g(V) \cap g(K): V \in \Sigma\}$ is a filterbase on Y. Consequently, $\theta\text{-adh}\,\mathcal{W} \neq \varnothing$. Furthermore, $\theta\text{-adh}\,\mathcal{W} \subset \{g(x)\} \cap \theta\text{-cl}\,[g(K)]$ by the properties of θ-closure and Theorem 2.2 above (since g has a strongly-closed graph).

Sufficiency. Let Y be Hausdorff, let $x_0 \in Y$ and suppose $\mathcal{W}$ is a filterbase on Y which does not θ-adhere to any point in $Y - \{x_0\}$. Let $X = Y$ be the space associated with x_0 and $\mathcal{W}$. X is in class $\mathbb{S}$ by Theorem 2.3. Let $i: X \to Y$ be the identity function. If $x \neq y$ and $x \neq x_0$, choose W open in Y about y with $x \notin \mathrm{cl}\,[W]$. Then $\{x\}$ is open in X and $(\{x\} \times \mathrm{cl}\,[W]) \cap G(i) = \varnothing$. If $x \neq y$ and $x = x_0$ then $y \neq x_0$, so there is an $F \in \mathcal{W}$ and W open about y satisfying $x_0 \notin \mathrm{cl}\,[W]$ and $F \cap \mathrm{cl}\,[W] = \varnothing$. $F \cup \{x_0\}$ is open in X and $((F \cup \{x_0\}) \times \mathrm{cl}\,[W]) \cap G(i) = \varnothing$. We have proved that i has a strongly-closed graph. Thus, i is weakly-continuous at x_0 and by Theorem 2.1 we conclude that $i(\mathrm{cl}\,[F]) \subset \theta\text{-cl}\,[F]$ for each $F \in \mathcal{W}$. Since $x_0 \in \mathrm{cl}\,[F]$ for each $F \in \mathcal{W}$, the proof is complete.

We move now to two of our main results.

3.2. THEOREM. *A Hausdorff space (X, τ) is H-closed if and only if for each topology τ^* on X with (X, τ^*) in class $\mathbb{S}$ for which the identity function $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed in X for each bijection $g: (X, \tau^*) \to (X, \tau)$ with a strongly-closed graph.*

PROOF. *Strong necessity.* Let (X, τ) be H-closed and let τ^* be any topology on X for which $i: (X, \tau^*) \to (X, \tau)$ has a strongly-closed graph. Let $g: (X, \tau^*) \to (X, \tau)$ be any function with a strongly-closed graph and let $v \in \mathrm{cl}\,[F(g)]$; g is weakly-continuous from Theorem 3.1. If $g(v) \neq v$, there are open sets $V \in \tau^*$ and $W \in \tau$ with $(v, g(v)) \in V \times W$ and $(V \times \mathrm{cl}\,[W]) \cap G(i) = \varnothing$. This derives from the fact that i has a strongly-closed graph. Since g is weakly-continuous, there is an $A \in \tau^*$ with $v \in A$ and $g(A) \subset \mathrm{cl}\,[W]$. $V \cap A \in \tau^*$ and $v \in V \cap A$; $g(V \cap A) \subset \mathrm{cl}\,[W]$, so there is no $x \in V \cap A$ satisfying $g(x) = x$. This contradiction establishes the necessity.

Sufficiency. Suppose $\mathcal{U}$ is a filterbase on (X, τ) which does not θ-adhere to any point in X. Choose $x_0 \in X$ and let τ^* be the topology on X associated with x_0 and $\mathcal{U}$. Using the same proof as that of the sufficiency of Theorem 3.1, (X, τ^*) is in class $\mathbb{S}$ and the identity function i: $(X, \tau^*) \to (X, \tau)$ has a strongly-closed graph. Choose $y_0 \in X - \{x_0\}$ and define g: $(X, \tau^*) \to (X, \tau)$ by $g(x_0) = y_0$, $g(y_0) = x_0$ and $g(x) = x$ otherwise; g is a bijection and we show that g has a strongly-closed graph. Let $(x,y) \in (X \times Y) - G(g)$. If $x \neq x_0$, choose $W \in \tau$ with $y \in W$ and $g(x) \notin \operatorname{cl}[W]$. Then $(\{x\} \times \operatorname{cl}[W]) \cap G(g) = \varnothing$. If $x = x_0$, $y \neq y_0$; so we may choose an $F \in \mathcal{U}$ and a W open about y satisfying

$$\{x_0\} \cup (\operatorname{cl}[W] \cap (F \cup \{x_0, y_0\})) = \{x_0\};$$

$$((F \cup \{x_0\}) \times \operatorname{cl}[W]) \cap G(g) = \varnothing.$$

This completes the demonstration that g has a strongly-closed graph. We see easily that $F(G) = X - \{x_0, y_0\}$ which is not τ^*-closed. This contradiction completes the proof.

3.3. THEOREM. *A Hausdorff space (X, τ) is H-closed if and only if for each topology τ^* on X with (X, τ^*) in class $\mathbb{S}$ for which the identity function i: $(X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g) = X$ whenever $F(g)$ is dense in X and the function g: $(X, \tau^*) \to (X, \tau)$ has a strongly-closed graph.*

PROOF. *Strong necessity.* In Theorem 3.2 we have found that for any topology τ^* on X for which the identity function i: $(X, \tau^*) \to (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed for any function g: $(X, \tau^*) \to (X, \tau)$ with a strongly-closed graph. So, if $F(g)$ is dense in (X, τ^*), we have $F(g) = X$.

Sufficiency. We follow the proof of the sufficiency of Theorem 3.2 to the point immediately preceding the definition of g. Choose $y_0 \in X - \{x_0\}$ and define g: $(X, \tau^*) \to (X, \tau)$ by $g(x) = x$ if $x \neq x_0$, and $g(x_0) = y_0$. Using an argument similar to that in the proof of the sufficiency of Theorem 3.2 we can see that g has a strongly-closed graph. Then $F(g) = X - \{x_0\}$ is dense in X, a contradiction which completes the proof.

4. **Minimal Hausdorff spaces.** See [1] for a survey of minimal topological spaces. In this paper we use the following characterization of minimal Hausdorff spaces as a primitive.

4.1. DEFINITION [3]. A Hausdorff space is *minimal Hausdorff* if each filterbase on the space with at most one θ-adherent point is convergent.

Theorem 7 of [3] provides that a function into a minimal Hausdorff space must be continuous if the function has a strongly-closed graph. In [5], it is proved that a weakly-continuous function into a Hausdorff space has a closed graph. The following easily established theorem is analogous to the result in [5] and enables us to see that if a space is minimal Hausdorff the class of continuous functions into the space coincides with the class of functions into the space with strongly-closed graphs.

4.1. THEOREM. *If Y is Hausdorff and $g: X \to Y$ is continuous, then g has a strongly-closed graph.*

4.2. THEOREM. *Let Y be minimal Hausdorff. Then $g: X \to Y$ is continuous if and only if g has a strongly-closed graph.*

In our last theorem and the final of our main results, we give a strong sufficiency to Theorem 7 of [3]; we also give a different proof of Theorem 7 than that in [3].

4.3. THEOREM. *A Hausdorff space Y is minimal Hausdorff if and only if for each space X in class $\mathbb{S}$, each bijection $g: X \to Y$ with a strongly-closed graph is continuous.*

PROOF. *Strong necessity* [3]. Let Y be minimal Hausdorff, let X be any space, let $g: X \to Y$ be any function with a strongly-closed graph and let $K \subset X$. Let $y \in g(\mathrm{cl}\,[K])$; choose $x \in \mathrm{cl}\,[K]$ with $g(x) = y$ and let Σ be an open set base at x. Then

$$\bigcap_{\Sigma} \theta\text{-cl}\,[g(V) \cap g(K)] = \{g(x)\}$$

since $\mathfrak{W} = \{g(V) \cap g(K): V \in \Sigma\}$ is a filterbase on Y, g has a strongly-closed graph and Y is H-closed. Since Y is minimal Hausdorff, we have $\mathfrak{W} \to y$. Thus, for any W open in Y about y, there is a $V \in \Sigma$ satisfying $g(V) \cap g(K) \subset W$. Consequently, $W \cap g(K) \neq \varnothing$ and $y \in \mathrm{cl}\,[g(K)]$.

Sufficiency. Let $\mathfrak{W}$ be a filterbase on Y with at most one θ-adherent point x_0. Let $X = Y$ be the space associated with x_0 and $\mathfrak{W}$, and let $i: X \to Y$ be the identity function. By means of the argument used in the proof of the sufficiency of Theorem 3.1, we see that i has a strongly-closed graph. Thus i is continuous and if W is open in Y about x_0, there is an $F \in \mathfrak{W}$ with $F \subset W$. Therefore, $\mathfrak{W} \to x_0$ and the proof is complete.

5. **Some examples.** In this section, we give some examples to indicate some limitations on the weakening of hypotheses in the theorems in this paper. By way of notation, we let N denote the set of positive integers. For each $k \in N$, we let $Z(k) = \{n \in N: n \geqslant k\}$, $E(k) = \{k + 1/2n: n \in N\}$, and $O(k) = \{k + 1/(2n - 1): n \in N\}$.

5.1. EXAMPLE. *The hypothesis cannot be weakened to "closed graph" in either Theorem* 3.1 *or Theorem* 4.3. Let

$$Y = \{-1, 0\} \cup \bigcup_{k=1}^{\infty} E(k) \cup \bigcup_{k=1}^{\infty} O(k) \cup N$$

with the topology generated by the following collection of sets as base: the subspace topology induced on $\bigcup_{k=1}^{\infty} E(k) \cup \bigcup_{k=1}^{\infty} O(k) \cup N$ by the usual topology of the reals along with the collection of all sets of the form $\{0\} \cup \bigcup_{k \geqslant m} E(k)$ and $\{-1\} \cup \bigcup_{k \geqslant m} O(k)$, where $m \in N$. Let $X = Y$ with the topology associated with 1 and the filterbase $\mathfrak{W} = \{Z(k): k \in N\}$. Y is minimal Hausdorff and X is in class $\mathcal{S}$. The identity function $i: X \to Y$ has a closed graph but is not weakly-continuous at $x = 1$. $G(i)$ is not strongly-closed since $(V \times \mathrm{cl}[W]) \cap G(i) \neq \varnothing$ for any V open about 1 and W open about 0.

5.2. EXAMPLE. *The hypothesis cannot be weakened to "the identity function $i: (X, \tau^*) \to (X, \tau)$ has a closed graph" in either Theorem* 3.2 *or Theorem* 3.3. Let $Y = \{0\} \cup \bigcup_{k=1}^{\infty} E(k) \cup N$ with the subspace topology from Y in Example 5.1. Let $X = Y$ be the space associated with 1 and the filterbase $\mathcal{W}$ in Example 5.1. Then X is in class $\mathcal{S}$, Y is H-closed and the identity function $i: X \to Y$ has a closed graph. Let $g: X \to Y$ be defined by $g(1) = 0$, $g(0) = 1$ and $g(x) = x$ otherwise. Then g is a bijection and has a strongly-closed graph. However $F(g) = X - \{0, 1\}$ is not closed in X. Now, let $h: X \to Y$ be defined by $h(x) = x$ if $x \neq 1$, and $h(1) = 0$. Then h has a strongly-closed graph and $F(h) = X - \{1\}$ which is dense in X.

5.3. EXAMPLE. *The hypothesis cannot be weakened to "$g: (X, \tau^*) \to (X, \tau)$ with a closed graph" in either Theorem* 3.2 *or Theorem* 3.3. Let Y be the space in Example 5.2 and let $X = Y$ be the space associated with 0 and the filterbase $\mathcal{W}$ from Example 5.1. The identity function $i: X \to Y$ has a strongly-closed graph. Let $g: X \to Y$ be defined by $g(0) = 1$, $g(1) = 0$ and $g(x) = x$ otherwise. Define $h: X \to Y$ by $h(x) = x$ if $x \neq 0$, and $h(0) = 1$. Then g and h have closed graphs which are not strongly-closed. $F(g) = X - \{0, 1\}$ which is not closed in X and $F(h) = X - \{0\}$ which is dense in X.

5.4. EXAMPLE. *"Weakly-continuous" cannot be replaced by "continuous" in Theorem* 3.1. In Example 5.2, the function $g: X \to Y$ is a bijection with a strongly-closed graph; g is not continuous at $x = 1$.

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