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INVARIANCE OF DOMAIN IN BANACH SPACES

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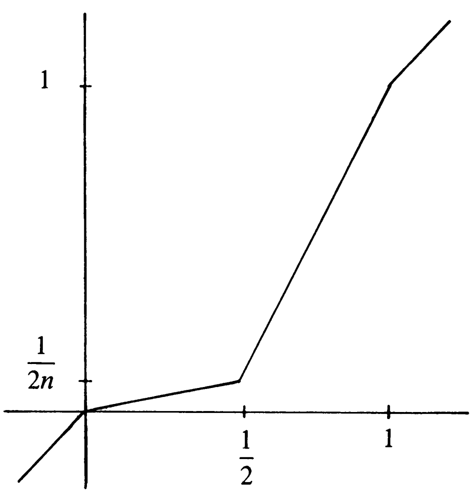
ABSTRACT. A continuous one-one map of the open unit disc of l^∞ onto itself which is not a homeomorphism is constructed.

The Brouwer theorem on invariance of domain states that *if G is an open subset of Euclidean space E and $f: G \to E$ is a continuous one-one map, then $f(G)$ is open and f is a homeomorphism*. This result has been extended to Banach spaces by Schauder [2] in the case when f is of the form $I + \phi$, ϕ being completely continuous, and by Tromba [3] in the case when f is a Fredholm map of index zero.

In general, invariance of domain fails in a Banach space since many Banach spaces are linearly homeomorphic to proper subspaces of themselves; furthermore Klee [1] has constructed a homeomorphism of separable Hilbert space onto a closed half-space. However, if the theorem is weakened so that $f(G)$ is already assumed to be open, does the result then hold in a Banach space? The answer is no as can be seen from the following extremely simple example.

Let g_n: $R \to R$ be the map with the graph



and let G: $l^\infty \to l^\infty$ be given by

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$$G(x_1, x_2, \ldots, x_n, \ldots) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n), \ldots).$$

Then G is clearly one-one and takes the open unit disc onto itself. Furthermore G is continuous, in fact Lipschitz continuous, since $\|G(x) - G(y)\| < 2\|x - y\|$. But G^{-1} is not continuous at the origin since $(1/n, 1/n, \ldots) \to 0$ and $\|G^{-1}(1/n, 1/n, \ldots)\| > 1/2$. Therefore G is a continuous one-one map of the open unit disc of l^∞ onto itself which is not a homeomorphism.

Also it is easy to see that G can be made C^∞ by properly rounding off corners. The derivative of G will then have the form

$$G'(x_1, x_2, \ldots)(\varepsilon_1, \varepsilon_2, \ldots) = (g_1'(x_1)\varepsilon_1, g_2'(x_2)\varepsilon_2, \ldots)$$

and will be a continuous transformation as long as $(g_1'(x_1), g_2'(x_2), \ldots) \in l^\infty$.

References

1. V. L. Klee, Jr., *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Soc. **74** (1953), 10–43. MR **14**. 989.

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3. A. J. Tromba, *Some theorems on Fredholm maps*, Proc. Amer. Math. Soc. **34** (1972), 578–585. MR **45** #7762.

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