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ON THE FRACTIONAL PARTS OF $n/j, j = o(n)$

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ABSTRACT. Dirichlet's result that if $J(n) = o(n)$ but $n^{1/2} = o(J(n))$, the numbers n/j for $j = 1, \ldots, J(n)$ are nearly uniformly distributed modulo 1 (with error $\to 0$ as $n \to \infty$) is extended, $n^{1/2}$ being replaced by n^α for any $\alpha > 0$.

1. To illustrate the problem considered here (and the results): for large n, the real numbers n/j for $j = 1, 2, \ldots, [n^{1/2}]$, reduced modulo 1, are nearly uniformly distributed. That is, for $t \in (0, 1)$, the fraction of those numbers n/j that lie between $[n/j]$ and $[n/j] + t$ differs from t by at most $\varepsilon(n)$, where $\varepsilon(n) \to 0$ as $n \to \infty$.

If $n^{1/2}$ is replaced by any function $J(n)$ satisfying $J(n) = o(n)$, but $n^{1/2} = o(J(n))$, the near-uniform distribution of those numbers is a result of Dirichlet (see [1, p. 327]), who showed also that the distribution is not uniform if $J(n) \neq o(n)$. This paper replaces Dirichlet's exponent $1/2$ by any $\alpha > 0$.

Much the hardest part of the proof is due to A. Walfisz, who proved a lemma on the distribution of some of the numbers in question. In 1932 Walfisz applied his lemma to estimates of the number of lattice points in an ellipsoid [2]; in 1963 he gave some other applications [3]. But this application, which seems the most natural one, also seems never to have been done.

THEOREM. *If $J(n) = o(n)$ but some n^α, $\alpha > 0$, is $o(J(n))$, then the fraction of the first $[J(n)]$ numbers n/j* (mod 1) *which lie in an interval of length t in R/Z differs from t by at most ε, where $\varepsilon \to 0$ as $n \to \infty$.*

We are indebted to Thomas Cusick and Lowell Schoenfeld for lively and helpful discussions.

2. Walfisz's lemma concerns sums of the complex numbers $e(n/j) = \exp(2\pi i n/j)$. The following specialization will suffice.

LEMMA (WALFISZ). *Let r be a positive integer, $w \in [0, 1]$, $R = 2^{r-1}$, $R_1 = R(r + 1)$, M between $n^{1/(r+2)}$ and $n^{2/(r+3)}$ (n need not be an integer). Then*

$$\sum_{j=M}^{j=2M} e(n/(j + w)) = O(M^{1-1/R-1/R_1} n^{1/R_1} \log n).$$

All we want of the concluding expression is that it is $o(M)$. Actually it is not, for the smallest M allowed, viz. $n^{1/(r+2)}$; there the estimate is

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$O(M \log M)$ (and is, of course, worthless). We narrow the requirements by adding

$$(*) \qquad M \geqslant n^{1 \cdot 1/(r+2)}.$$

Still the different values of r give overlapping intervals from $2n^{1/2}$ down through all N^α in which the average value of $M + 1$ successive terms $e(n/(j + w))$ beginning at $j = M$ is always (as a function of n) $o(1)$. More precisely, for each r, if n is large enough, each of those averages is less than ε (by Walfisz's proof and $(*)$).

$2n^{1/2}$ is not big enough (to adjoin Dirichlet's case). However, the $o(1)$ conclusion extends all the way up $o(n)$.

COROLLARY. *For $M(n) \geqslant n^{1 \cdot 1/3}$, $M(n) = o(n)$, the average of $e(n/j)$ as j goes from $M(n)$ to $2M(n)$ is $o(1)$.*

PROOF. First, if we replace some n by $n' = n/M(n)$, n' will still go to infinity with n and "$o(1)$" may be referred equally well to varying n or varying n'. This still applies though, precisely, we introduce $b(n) = 1 + [M(n)^2/n]$ and put $n^* = n/b(n)$, so that $n/j = n^*/(j/b(n))$. Unless $M(n) \geqslant n^{1/2}$, $b(n) = 1$ and we did nothing. Otherwise (with negligible error) we replace M by $M^* = (n^*)^{1/2}$. Precisely, add at most $b(n) - 1$ terms to the sequence to make its length a multiple of $b(n)$ (affecting the average by less than $b(n)/M(n) = o(1)$). The sequence of expressions $e(n^*/(j/b(n)))$ decomposes into $b(n)$ sequences in which denominators form progressions with difference 1; each has, with error $o(1)$, the form in the lemma. So each has average $o(1)$, and the average of those $b(n)$ averages is still $o(1)$.

3. We wish to apply Weyl's criterion, familiar in this form: a sequence (a_i) of complex numbers of modulus 1 is uniformly distributed if, for $k = 1, 2, \ldots$, the average of the first n kth powers a_i^k approaches 0 as $n \to \infty$. We need the following form.

LEMMA (WEYL). *For each $\varepsilon > 0$ there exists N such that given a finite family of complex numbers of modulus 1, if for $k \leqslant N$ the average of a_i^k has modulus less than $1/N$, then the fraction of the a_i which lie in an interval on the unit circle of length $2\pi t$ is between $t - \varepsilon$ and $t + \varepsilon$.*

Supposing this false, for some ε we should have a sequence of examples, $N \to \infty$, missing by ε on certain intervals. For a subsequence, the intervals converge to a limit, and it is simple routine to patch together an infinite sequence (a_i) violating the criterion as previously stated, which is absurd.

The theorem follows. For, first, for $M \geqslant n^\alpha$ but $o(n)$, the average of $e(n/j)$ for j from M to $2M$ is small (for large n). Also M exceeds $(2n)^{\alpha/2}$, and the average of $e(2n/j)$ for those j is small. This is true of $e(kn/j)$, uniformly in k, as long as $k \leqslant n$, $(kn)^{\alpha/2} \leqslant n^\alpha$. So the modified Weyl criterion tells us that those $e(n/j)$ are uniformly distributed to within ε ($\varepsilon \to 0$ as $n \to \infty$). To within 2ε, we can apply this to all j less than $J(n)$, for $J(n) \geqslant 2^s n^\alpha$ but $J(n) = o(n)$. Let $M_0 = [2^{-s}J(n)]$, $M_i = 2^i M_0$ for $i < s$; the s intervals $[M_i, 2M_i]$ reach from 1 to $J(n)$, with negligible error and negligible overlap, and all have $e(n/j)$ uniformly distributed (within ε).

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