PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

NEW PROOF OF A DENSITY THEOREM FOR THE BOUNDARY OF A CLOSED SET

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ABSTRACT. From Browder [1] the following theorem is known: *Let F be a closed subset of the Banach space E; then the set R of points $x \in \partial F$, such that $F \cap C = \{x\}$ for at least one convex C with nonempty interior, is dense in ∂F.* A proof of this will be given by means of a theorem of Martin [4] on ordinary differential equations.

Proofs of the just quoted result have been given by Browder [1], [2], Daneš [3], and Phelps [5]. A completely different proof runs as follows: Assume the statement of Browder's theorem to be false. Then there exists a point $p \in \partial F$ and an open, convex neighbourhood U of p such that

$$R \cap \partial F \cap U = \varnothing. \tag{1}$$

Now, for every function $f\colon U \to E$ the formula

$$\liminf_{h\to 0+} \frac{1}{h}\,|F, x + hf(x)| = 0 \qquad (x \in \partial F \cap U) \tag{2}$$

is valid (see below; $|F, y|$ denotes the distance from F to the point y). Choose $q \in U \setminus F$ and define

$$f(x) \equiv q - p \qquad (x \in U).$$

Then the unique solution of the initial value problem

$$u(0) = p, \quad u'(t) = f(u(t)) \qquad (0 \leqslant t \leqslant 1)$$

is $u(t) = (1 - t)p + tq$, but since f is Lipschitz-continuous and (2) holds, $u(t)$ must remain in F by Theorem 4 of Martin [4]. This yields $q = u(1) \in F$, leading to a contradiction.

To prove (2), fix $x \in \partial F \cap U$ and let ε, $h_0 > 0$. Then

$$C_{\varepsilon,h_0} = \{x + hf(x) + h\varepsilon s \mid 0 \leqslant h \leqslant h_0, \|s\| \leqslant 1\}$$

is a convex set with nonempty interior. By (1), $x \notin R$, and so there is some $y \in F \cap C_{\varepsilon,h_0}$, $y \neq x$, i.e.

$$y = x + hf(x) + h\varepsilon s, \quad \text{where } 0 < h \leqslant h_0, \ \|s\| \leqslant 1.$$

Hence

$$|F, x + hf(x)| \leqslant \|x + hf(x) - y\| = h\varepsilon\|s\| \leqslant h\varepsilon,$$

yielding

Received by the editors February 9, 1976.
AMS (MOS) subject classifications (1970). Primary 46B99; Secondary 34G05.

$$(1/h)|F, x + hf(x)| \leqslant \varepsilon \quad \text{for some } h \in (0, h_0].$$

Since ε and h_0 have been chosen arbitrarily, (2) is established.

ACKNOWLEDGEMENT. The author is deeply indebted to Professor R. M. Redheffer for his encouragement to publish this note.

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