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ON FIXED-POINT FREE FIBREWISE MAPS OF PROJECTIVE BUNDLES

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ABSTRACT. Let E be a vector bundle and $P(E)$ its associated projective bundle. Some necessary conditions on the characteristic classes of E for existence of a fibrewise fixed-point free map $P(E) \to P(E)$ are obtained.

1. **Introduction.** Let $E \to B$ be an F-vector bundle, where F denotes either the reals $\mathbf{R}$, the complex numbers $\mathbf{C}$, or the quaternions $\mathbf{H}$, and let $p\colon P(E) \to B$ denote the associated projective bundle. We consider the question of existence of fibrewise maps $f\colon P(E) \to P(E)$, i.e. continuous maps such that

$$\begin{tikzcd} P(E) \arrow[rr, "f"] \arrow[dr, "p"'] & & P(E) \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

commutes, which are fixed-point free. By consideration of individual fibres, it follows, from the Lefschetz fixed point theorem, that a necessary condition for the existence of such an f is that the fibre dimension of E be even over F. The main purpose of this paper is to obtain some necessary conditions on the characteristic classes of E for existence of such a map f. For example, we show that if the base space B is simply-connected, the existence of a fibrewise fixed-point free $f\colon P(E) \to P(E)$ implies that all odd Stiefel-Whitney classes of E vanish (Corollary 3.3). Similar results are obtained in the complex and quaternionic cases for the Chern classes and symplectic Pontrjagin classes, respectively.

Existence of a fibrewise fixed-point free map $f\colon P(E) \to P(E)$ is shown, in §2, to be equivalent to the existence of what we call a PA-structure on E, in analogy with the A-structures of James (see [1]–[3]). It is easily seen that the existence of an equivariant A-structure on E, in the sense of [1], implies the existence of a PA-structure on E, but we show in §2 that existence of a PA-structure does not imply existence of an A-structure.

The characteristic class considerations are carried out in §3, the main tool

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being the structure of the cohomology ring of $P(E)$ in terms of characteristic classes of E.

Some examples are given in §4. In particular, it is shown in 4.1 that the Whitney sum of two vector bundles admitting PA-structures need not admit a PA-structure. This is in contrast to the situation for A-structures [3, Theorem 1.4].

2. **PA-structures.** If U is an F-vector space, we denote by $P(U)$ the projective space of U, i.e. the space of one-dimensional F-subspaces of U. If $u \in U - \{0\}$, let $[u] \in P(U)$ denote the one-dimensional subspace spanned by u. Assume that U has an inner product (euclidean, hermitian, or symplectic depending on whether $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$). Let $S(U)$ denote the unit sphere in U, $V(U) = \{(u_1, u_2) \in S(U) \times S(U): u_1 \perp u_2\}$ = Stiefel manifold of orthonormal 2-frames of U, and $Z(U) = \{([u_1], [u_2]) \in P(U) \times P(U): u_1 \perp u_2\}$.

Let $E \to B$ be an F-vector bundle of dimension $2n$ over F, with a metric (i.e. the structural group is $O(2n)$, $U(2n)$, or $Sp(2n)$ depending on whether $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$). Form the associated bundles $p: P(E) \to B$, $S(E) \to B$, $V(E) \to B$, $Z(E) \to B$ with fibres $P(F^{2n})$, $S(F^{2n})$, $V(F^{2n})$, $Z(F^{2n})$, respectively. If $x \in B$ and E_x denotes the fibre over x in E, then the fibres over x in these associated bundles are canonically identified with $P(E_x)$, $S(E_x)$, $V(E_x)$, $Z(E_x)$, respectively. Moreover we have fibre bundles $q: Z(E) \to P(E)$ and $r: V(E) \to S(E)$ given by $q([u], [v]) = [u]$ and $r(u, v) = u$. The fibres of these bundles are $P(F^{2n-1})$ and $S(F^{2n-1})$, respectively. In [1], an *A-structure* on E is defined to be a section of $V(E) \to S(E)$.

DEFINITION 2.1. A *PA-structure* on E is a section of the bundle $q: Z(E) \to P(E)$.

PROPOSITION 2.2. *There exists a fibrewise fixed-point free map $f: P(E) \to P(E)$ if and only if E admits a PA-structure.*

PROOF. If $s: P(E) \to Z(E)$ is a PA-structure on E, define $f: P(E) \to P(E)$ by $f[u] = \pi s[u]$ where $\pi: Z(E) \to P(E)$ is given by $\pi([u], [v]) = [v]$. Then f is fibrewise and fixed-point free.

Conversely, if $f: P(E) \to P(E)$ is a fibrewise fixed-point free map, define $s: P(E) \to Z(E)$ as follows: for $[u] \in P(E_x)$, let $s[u] = ([u], \pi_u f[u])$ where $\pi_u: E_x \to E_x$ is orthogonal projection on $u^{\perp}$. Then s is a PA-structure on E.

PROPOSITION 2.3. *If $E \to B$ admits a PA-structure, and $f: X \to B$ is any continuous map, then $f^* E \to X$ admits a PA-structure.*

PROOF. $P(f^*E) = \{(x, y) \in X \times P(E): y \in P(E_{f(x)})\}$. If $g: P(E) \to P(E)$ is a fibrewise fixed-point free map, then so is $h: P(f^*E) \to P(f^*E)$ given by $h(x, y) = (x, g(y))$.

In [1], an *equivariant A-structure* on E is defined to be an A-structure $s: S(E) \to V(E)$ such that $s(uz) = s(u)z$ for all $u \in S(E)$, $z \in S(F)$, and it is shown there that such cannot exist unless $F = \mathbf{R}$. An equivariant A-structure on E yields a PA-structure on E by passage to quotients.

Note that if E is an F 2-plane bundle, then the fibre of $Z(E) \to P(E)$ is a single point, and so a unique PA-structure exists for E. However, by [2, Theorem 1.2], if E admits an A-structure, then all odd Stiefel-Whitney classes of E vanish and, in particular, E must be orientable. Thus real nonorientable 2-plane bundles are examples of vector bundles which admit PA-structures, but not A-structures.

3. **Characteristic classes.** Throughout this section d will denote the dimension of F over $\mathbf{R}$, and the coefficients for cohomology will be understood to be $\mathbf{Z}_2$ if $F = \mathbf{R}$, and $\mathbf{Z}$ if $F = \mathbf{C}$ or $\mathbf{H}$.

Let $E \to B$ be an F vector bundle with a metric. Let $L(E) \to P(E)$ denote the canonical line bundle over $P(E)$, i.e. the fibre over $[u]$ consists of the points on the line $[u]$. Then $p^*E \cong L(E) \oplus L(E)^{\perp}$, where $p\colon P(E) \to B$ denotes the projection, and we can identify $Z(E)$ with $P(L(E)^{\perp})$.

Let $\sigma_i(E) \in H^{di}(B)$ denote the ith characteristic class of E (Stiefel-Whitney, Chern, or symplectic Pontrjagin depending on whether $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$). As is well known, the structure of $H^*(P(E))$ is as follows:

3.1. $H^*(P(E))$ is a free $H^*(B)$-module with basis $1, x, x^2, \dots, x^{\dim E - 1}$ where the module structure is via p^*, and $x = \sigma_1(L(E))$. The multiplicative structure is determined by the relation

$$\sum_{i=0}^{\dim E} (-1)^i p^*\sigma_i(E) x^{\dim E - i} = 0.$$

For an exposition see, e.g., [4, Chapter V]. We follow the sign conventions of [4].

THEOREM 3.2. *Let $E \to B$ be an F $2n$-plane bundle with a metric. Suppose there exists a fibrewise fixed-point free map $P(E) \to P(E)$. Then there exists a class $a \in H^d(B)$ such that*

$$\sum_{k=0}^{2n-1-r} \sum_{i=k}^{r+k} (-1)^i \binom{i}{k} \sigma_{2n-1-r-k}(E) a^k = 0$$

for $0 \leqslant r \leqslant 2n - 2$.

PROOF. By 2.2 there exists a section $s\colon P(E) \to Z(E) = P(L(E)^{\perp})$ of $q\colon P(L(E)^{\perp}) \to P(E)$. By 3.1, $H^*(P(L(E)^{\perp}))$ is the free $H^*(P(E))$-module (via q^*) on $1, x, x^2, \dots, x^{2n-2}$, where $x = \sigma_1(L(L(E)^{\perp}))$, and we have the relation

$$\sum_{i=0}^{2n-1} (-1)^{2n-1-i} q^*\sigma_{2n-1-i}\big(L(E)^{\perp}\big) x^i = 0. \tag{1}$$

Since $p^*(E) \cong L(E) \oplus L(E)^{\perp}$, it follows from the Whitney product formula that

$$p^* \sum_{i=0}^{2n} \sigma_i(E) = (1 + y) \sum_{i=0}^{2n-1} \sigma_i\big(L(E)^{\perp}\big) \tag{2}$$

where $y = \sigma_1(L(E))$. From (2) it follows that

$$\sigma_k\big(L(E)^{\perp}\big) = \sum_{j=0}^{k} (-1)^j y^j p^* \sigma_{k-j}(E) \quad \text{for } 0 \leqslant k \leqslant 2n-1. \tag{3}$$

Substituting into (1) and factoring out $(-1)^{2n-1}$ we obtain

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1-i} (-1)^{i+j} q^* y^j q^* p^* \sigma_{2n-1-i-j}(E) x^i = 0. \tag{4}$$

Applying s^* to (4) we obtain

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1-i} (-1)^{i+j} y^j p^* \sigma_{2n-1-i-j}(E)(s^*x)^i = 0. \tag{5}$$

By 3.1 there exist unique classes $a \in H^d(B)$, $b \in H^0(B)$ such that $s^*x = p^*a + (p^*b)y$. Substituting into (5) we obtain

$$\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1-i} \sum_{k=0}^{i} (-1)^{i+j} \binom{i}{k} p^*\big[\sigma_{2n-1-i-j}(E) a^k b^{i-k}\big] y^{i+j-k} = 0. \tag{6}$$

Setting $r = i + j - k$ and changing the order of summation we obtain

$$\sum_{r=0}^{2n-1} \sum_{k=0}^{2n-1-r} \sum_{i=k}^{r+k} (-1)^{r+k} \binom{i}{k} p^*\big[\sigma_{2n-1-r-k}(E) a^k b^{i-k}\big] y^r = 0. \tag{7}$$

From (7) and the fact (3.1) that $1, y, y^2, \ldots, y^{2n-1}$ form a free module basis of $H^*(P(E))$ over $H^*(B)$, we have

$$\sum_{k=0}^{2n-1-r} \sum_{i=k}^{r+k} (-1)^k \binom{i}{k} \sigma_{2n-1-r-k}(E) a^k b^{i-k} = 0 \tag{8}$$

for $0 \leqslant r \leqslant 2n-1$. Taking $r = 2n-1$ in (8), we obtain $1 + \sum_{i=1}^{2n-1} b^i = 0$, from which it follows that $b = -1$. Substituting this into (8) yields the theorem.

COROLLARY 3.3. *If $E \to B$ is as in* 3.2, *and if $H^d(B) = 0$, then $\sigma_i(E) = 0$ for i odd.*

PROOF. For then $a = 0$, and so only the $a^0 = 1$ terms in 3.2 survive, yielding, for $0 \leqslant r \leqslant 2n-2$, $0 = \sum_{i=0}^{r}(-1)^i \sigma_{2n-1-r}(E)$. For r even, this yields $\sigma_{2n-1-r}(E) = 0$.

COROLLARY 3.4. *Let $E \to B$ be as in* 3.2. *Then $\sigma_1(E)$ is divisible by n. In particular, if $F = \mathbf{R}$ and n is even, E must be orientable.*

PROOF. Setting $r = 2n - 2$ in 3.2 yields $\sigma_1(E) = na$. In particular, if $F = \mathbf{R}$ and n is even we obtain $w_1(E) = 0$.

4. Examples.

EXAMPLE 4.1. Let L denote the canonical F line bundle over $P(F^m)$, $m \geqslant 2$, and L_0 the trivial line bundle over $P(F^m)$. Let π_i: $P(F^m) \times P(F^m) \to P(F^m)$, $i = 1, 2$, denote the projections on the first and second factors, respectively. Let $E_i = \pi_i^*(L \oplus L_0)$, $i = 1, 2$. Since E_1 and E_2 are both 2-plane bundles, they both admit PA-structures (§2). Write $u = \sigma_1(L)$. By the Whitney product formula, it follows that $\sigma_1(E_1 \oplus E_2) = u \times 1 + 1 \times u$, which is not divisible by 2 in $H^d(P(F^m) \times P(F^m))$. Thus by 3.4, $E_1 \oplus E_2$ does not admit a PA-structure. Thus the collection of F vector bundles admitting PA-structures is not closed under Whitney sum.

EXAMPLE 4.2. Let $E \to B$ be any real vector bundle. Then there exists a fibrewise fixed-point free map f: $P(E \oplus E) \to P(E \oplus E)$ given by $f[u, v] = [v, -u]$.

EXAMPLE 4.3. Suppose $F = \mathbf{C}$ or $\mathbf{H}$, and let L and L_0 be as in 4.1. Let $E = L \oplus L_0 \oplus L_0$. Then $\sigma_1(E \oplus E) = 2u$, which is not divisible by 3 in $H^d(P(F^m))$. Thus, by 3.4, $E \oplus E$ does not admit a PA-structure. Thus the analogue of 4.2 for the complex and quaternionic cases is false.

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