

CLASSIFICATION OF SPACES OF THE SAME n -TYPE FOR ALL n

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ABSTRACT. The set of homotopy equivalence classes of CW-spaces Y for which the Postnikov approximations are homotopy equivalent to those of a given space X is shown to be in one-to-one correspondence with \varprojlim^1 of a tower of homotopy automorphism groups.

Recall that two CW-spaces X and Y are said to have the same n -type if the n th Postnikov approximations $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent. The homotopy classification problem for infinite dimensional complexes can be studied in two steps. First, classify all finite dimensional complexes (!) and then classify for a given X all complexes of the same n -type for all n as X . The purpose of this note is to codify this second step.

THEOREM I. *Let X be a connected CW-space. Denote by $SNT(X)$ the set of homotopy equivalence classes of CW-spaces Y such that Y has the same n -type for all n as X . Then $SNT(X)$ is in one-to-one correspondence with the pointed set $\varprojlim^1(\text{Aut}(X^{(n)}))$, where $\text{Aut}(X^{(n)})$ is the group of homotopy classes of homotopy equivalences of $X^{(n)}$.*

The relevance of \varprojlim^1 -type arguments is clear from the work of Adams [1] and Gray [4], but Theorem I seems to be the first explicit formulation of the classification problem. The first application is to study some cases for which $SNT(X) = \{X\}$.

COROLLARY II. *Let X be a simply connected CW-space such that $\pi_i(X)$ is finitely generated in each dimension.*

- (a) (Adams [1]) *If $\pi_i(X)$ is finite for all $i > 0$, then $SNT(X) = \{X\}$.*
- (b) *If X_0 denotes the rationalization of X [8], [3], [5], then $SNT(X_0) = \{X_0\}$.*
- (c) *If \hat{X} denotes the profinite completion of X [8], then $SNT(\hat{X}) = \{\hat{X}\}$.*

The proof of the corollary is the demonstration that for the towers of automorphism groups involved, the \varprojlim^1 set reduces to the one point set.

PROPOSITION III. *Let X be a simply connected CW-space such that $\pi_i(X)$ is finitely generated in each dimension. Then $\varprojlim^1(\text{Aut}(X^{(n)})) \approx \varprojlim^1(\text{Im}(n))$, where $\text{Im}(n)$ denotes $\{\text{image}(\text{Aut}(X^{(n)}) \rightarrow \text{Aut}(X_0^{(n)}))\}$.*

Finally, we study in some detail an example of a rational H -space X such that $SNT(X_p) \neq \{X_p\}$ for any prime p . This example is motivated by Gray's proof [4] that for $W = \mathbb{C}P^\infty \times \Omega S^3$, $SNT(W) \neq \{W\}$. Our example shows

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that the finite dimensionality hypothesis of Theorem 1.1 of [2] is necessary: there the result is that if Y' and Y'' are simply connected finite dimensional rational H -spaces such that $\hat{Y}'_p \simeq \hat{Y}''_p$, then $Y'_p \simeq Y''_p$. However, Corollary II(c) and the example above show that this cannot hold in general.

We note that the classification argument of §1 is a variant of the standard "principal homogeneous space" argument of algebraic geometry, and can be modified to classify the possible pullbacks of other diagrams of spaces in terms of the automorphism groups, e.g., §3 of Wilkerson [10]. I would like to acknowledge helpful discussions with J. F. Adams, A. K. Bousfield, J. Cohen, G. Porter and P. Trauber on this material.

I. The classification. We first recall the definition of the \varprojlim^1 functor for a tower of groups (G_n, q_n) , where $q_n: G_n \rightarrow G_{n-1}$: there is an action of the product group $\prod_n G_n$ on the product set $\prod_n G_n$, defined as $(\gamma_n)(\alpha_n) = (\gamma_n \alpha_n (q_{n+1}(\gamma_{n+1})^{-1}))$. The orbit set of this action is denoted as $\varprojlim^1 G_n$ and has as a base point the equivalence class of (e_n) , where e_n is the identity element of G_n .

For the proof of Theorem I, the simplicial setting offers some technical advantages. The Moore-Postnikov decomposition is functorial and inverse limits can be taken as inverse limits of sets. X is to be considered a pointed connected minimal Kan simplicial set in the following. We first recall the Moore-Postnikov decomposition of X , see, for example, May [6]. Define an equivalence relation on the simplices of X , $x \sim^n y$, if and only if all corresponding faces of x and y of dimension less than or equal to n agree. $X^{(n)}$, the quotient of X by \sim^n , is again a pointed connected minimal Kan complex. $X \rightarrow X^{(n)}$ is a Kan fibration, and the triangle

$$\begin{array}{ccc} X & \longrightarrow & X^{(n)} \\ & \searrow & \downarrow p_n \\ & & X^{(n-1)} = (X^{(n)})^{(n-1)} \end{array}$$

commutes. If \varprojlim is applied degree-wise to the tower of fibrations $(X^{(n)}, p_n)$, then $X = \varprojlim (X^{(n)}, p_n)$ as simplicial sets. It is also clear from the construction that any simplicial map $f: X \rightarrow Y$ induces $f_n: X^{(n)} \rightarrow Y^{(n)}$ for all n , which commutes with the projections, and conversely, a family of maps f_n which commutes with the projections determines a unique map $f: X \rightarrow Y$.

Note that if X and Y are minimal complexes, then any homotopy equivalence $f: X \rightarrow Y$ is actually a simplicial isomorphism. This will be convenient for technical reasons. Define $\text{Simp Aut}(X^{(n)})$ to be the set of all simplicial isomorphisms of $X^{(n)}$ to itself. We begin the classification of $SNT(X)$ by defining a set map $\varphi: \prod_n \text{Simp Aut}(X^{(n)}) \rightarrow SNT(X)$. Let $\bar{\alpha} = (\alpha_0, \alpha_1, \dots)$ belong to $\prod_n \text{Simp Aut}(X^{(n)})$. Define

$$\varphi(\bar{\alpha}) = X_{\bar{\alpha}} = \varprojlim (X^{(n)}, \alpha_{n-1} p_n),$$

where the inverse limit is applied degree-wise. Then $X_{\bar{\alpha}}$ is again a minimal Kan complex and belongs to $SNT(X)$. The homotopy lifting property of the fibrations $\alpha_{n-1} p_n: X^{(n)} \rightarrow X^{(n-1)}$ shows that if α_n is homotopic to α'_n for all n ,

then X'_α is homotopy equivalent to $X_{\bar{\alpha}}$. Thus $\varphi: \prod_n \text{Aut}(X^{(n)}) \rightarrow \text{SNT}(X)$ is well defined.

We show now that φ is onto. Let Y be a minimal Kan complex in $\text{SNT}(X)$. Choose for each n a homotopy equivalence (simplicial isomorphism) $g_n: Y^{(n)} \rightarrow X^{(n)}$. Denote the map induced by g_{n+1} as $(g_{n+1})_n: Y^{(n)} \rightarrow X^{(n)}$ and define $(\alpha_n) = ((g_n)(g_{n+1})_n^{-1})$. Then $Y = X_{\bar{\alpha}} = \lim(X_n, \alpha_{n-1}p_n)$ as simplicial sets. That is, φ is surjective.

We now show that φ remains well defined after passage to the quotient set $\varprojlim^1(\text{Aut}(X^{(n)}))$; that is, if $\bar{\alpha} = \bar{\gamma} \circ \bar{\beta}$, then $X_{\bar{\alpha}} \simeq X_{\bar{\beta}}$. The square below commutes up to homotopy:

$$\begin{array}{ccc}
 X^{(n)} & \xrightarrow{\quad g_n = \gamma_n \quad} & X^{(n)} \\
 \downarrow p_n & & \downarrow p_n \\
 X^{(n-1)} & & X^{(n-1)} \\
 \downarrow \beta_{n-1} & & \downarrow \alpha_{n-1} \\
 X^{(n-1)} & \xrightarrow{\quad g_{n-1} = \gamma_{n-1} \quad} & X^{(n-1)}
 \end{array}$$

We wish to find a map homotopic to γ_n such that the diagram commutes. Assume this has been done at all levels below n , and that the $\gamma_m = g_m$ in the diagram are the modified maps, for $m < n$. Then the covering homotopy property of the fibrations shows that g_n homotopic to the original γ_n exists such that the diagram commutes. Thus, there is an isomorphism $g: X_{\bar{\beta}} \rightarrow X_{\bar{\alpha}}$. Conversely, given an isomorphism $g: X_{\bar{\beta}} \rightarrow X_{\bar{\alpha}}$, define $\gamma_n = g_n$ and it is easy to see that $\bar{\alpha} = \bar{\gamma} \circ \bar{\beta}$. Thus $\varphi: \varprojlim^1(\text{Aut}(X^{(n)})) \rightarrow \text{SNT}(X)$ is monic. That is φ is an isomorphism of pointed sets, since it takes the equivalence class of (e_n) to $\{X\}$, and is monic and surjective.

The proof for the case that X' is a CW-complex now follows by taking the X above to be a minimal subcomplex of the singular complex of X' and observing that the geometric realization of $X^{(n)}$ gives a Postnikov tower for X' , up to homotopy.

II. Vanishing of \varprojlim^1 . Recall that a tower of groups (G_n, q_n) is Mittag-Leffler if for each n , there exists N_n such that $\text{image } q_{n+m} \cdots q_{n+1} = \text{image } q_{n+N_n} \cdots q_{n+1}$ for all $m > N_n$. If (G_n, q_n) is Mittag-Leffler, then $\varprojlim^1 G_n = *$; see Bousfield and Kan [3]. Therefore Corollary II(b) follows directly from Lemma 2.1.

LEMMA 2.1. *If X is a simply connected CW-space such that $\pi_i(X) \otimes \mathbf{Q}$ is finitely generated over \mathbf{Q} for each i , then the tower of groups $(\text{Aut}(X_0^{(n)}), q_n)$ is Mittag-Leffler.*

PROOF. By Wilkerson [10] or Sullivan [9], there is a finitely generated \mathbf{Q} -Hopf-algebra $S(n)$ such that $\text{Aut}(X_0^{(n)}) = \text{Hom}_{\mathbf{Q}\text{-alg}}[S(n), \mathbf{Q}]$ and $p_n: X_0^{(n)} \rightarrow X_0^{(n-1)}$ induces $\tilde{p}_n: S(n-1) \rightarrow S(n)$ a map of \mathbf{Q} -algebras. That is,

$\text{Aut}(X_0^{(n)})$ is a linear algebraic group over \mathbf{Q} , and q_n is a morphism of linear algebraic groups. Thus the image of each $\text{Aut}(X_0^{(n+k)})$ is a closed subgroup of $\text{Aut}(X_0^{(n)})$ in the Zariski topology. That is, there is an ideal $I_{k,n}$ of $S(n)$ such that the image is exactly the elements of $\text{Hom}_{\mathbf{Q}\text{-alg}}[S(n), \mathbf{Q}]$ which annihilate $I_{k,n}$. Since $I_{k,n}$ is properly contained in $I_{k+1,n}$ if the image subgroups do not coincide, the noetherian property of $S(n)$ shows that $\text{Aut}(X_0^{(n)})$ is Mittag-Leffler.

LEMMA 2.2. *If (G_n, q_n) is a tower of compact topological groups and continuous homomorphisms, then $\varprojlim^1 G_n = *$.*

PROOF. This is an easy application of the finite intersection characterization of compactness and is well known.

LEMMA 2.3. *Let X be a simply connected CW-complex such that $\pi_*(X)$ is finitely generated over some subring of \mathbf{Q} . Then $\text{Aut}(\hat{X}_p^{(n)})$ is compact and $q_n: \text{Aut}(\hat{X}_p^{(n)}) \rightarrow \text{Aut}(\hat{X}_p^{(n-1)})$ is continuous.*

PROOF. This is similar to arguments in Sullivan [8]. By Bousfield and Kan [3], $\hat{X}_p^{(n)} \simeq (X^{(n)})_p^\wedge \simeq R_\infty X^{(n)}$ for $R = \mathbf{Z}/p\mathbf{Z}$. $R_\infty X^{(n)}$ is the inverse limit of a functorial tower of fibrations $R_s X^{(n)}$, where for $R = \mathbf{Z}/p\mathbf{Z}$, the homotopy groups of each $R_s X^{(n)}$ are finite in each dimension. Thus $\text{Aut}(R_s X^{(n)})$ is finite for all s . Now given $f: R_\infty Y \rightarrow R_\infty Y$, there is induced $R_s(f): R_s R_\infty Y \rightarrow R_s R_\infty Y$. However, the natural map $\phi: Y \rightarrow R_\infty Y$ induces $R_s \phi: R_s Y \rightarrow R_s R_\infty Y$ which is a homotopy equivalence for good spaces Y . That is, there is a natural map from $\text{Aut}(R_\infty X^{(n)})$ to the tower of groups $\text{Aut}(R_s X^{(n)})$. The finiteness of $[SX^{(n)}, R_s X^{(n)}]$ implies that $\varprojlim_s [X^{(n)}, R_s X^{(n)}] \rightarrow [X^{(n)}, R_\infty X^{(n)}]$ is a bijection, by the short exact sequence of [3] or Quillen [7] for the homotopy groups of the inverse limit of a tower of fibrations. Since $R_s X^{(n)}$ and $R_\infty X^{(n)}$ are \mathbf{Z}/p -complete [3], this holds with $X^{(n)}$ replaced by $R_\infty X^{(n)}$.

Thus $\text{Aut}(R_\infty X^{(n)}) \rightarrow \varprojlim \text{Aut}(R_s X^{(n)})$ is monic. Since $R_\infty X^{(n)} = \varprojlim R_s X^{(n)}$, it is onto. Therefore $\text{Aut}(R_\infty X^{(n)})$ is compact, since it is the inverse limit of a tower of finite groups, and the maps q_n are continuous since they are induced from families of compatible maps between the finite groups. This completes the proof of Corollary II(c), since $\text{Aut}(\hat{Y}) = \prod_p \text{Aut}(\hat{Y}_p)$.

III. **An example.** We first recall a special case of a theorem of Hilton, Mislin and Roitberg [5].

LEMMA 3.1. *Let X be a simply connected CW-complex with $\pi_*(X)$ finitely generated over some subring of \mathbf{Q} . Then $\text{Aut}(X^{(n)}) \rightarrow \text{Aut}(X_0^{(n)})$ has finite kernel K_n for all $n > 0$.*

Since \varprojlim^1 of a tower of finite groups is the one point set, the tailend of the six term exact sequence associated to the short exact sequence of towers of groups [3],

$$\{1\} \rightarrow \{K_n\} \rightarrow \{\text{Aut}(X^{(n)})\} \rightarrow \{\text{Im}(n)\} \rightarrow \{1\},$$

degenerates to $* \rightarrow \varprojlim^1 \text{Aut}(X^{(n)}) \xrightarrow{\psi} \varprojlim^1 \text{Im}(n) \rightarrow *$, an exact sequence of pointed sets. While this proves that ψ is surjective, it only states that the basepoint is taken to the basepoint by ψ . It remains then to show that if

$\psi(\bar{\beta}) \sim \psi(\bar{\alpha})$, then actually $\bar{\beta} \sim \bar{\alpha}$ in $\varprojlim^1 \text{Aut}(X^{(n)})$. By the next lemma, this is a general fact about certain short exact sequences of towers of groups.

LEMMA 3.2. *If $\{1\} \rightarrow \{G'_n\} \rightarrow \{G_n\} \xrightarrow{\psi} \{G''_n\} \rightarrow \{1\}$ is an exact sequence of towers of groups, then if $\{G'_n\}$ is Mittag-Leffler, $\varprojlim^1(\psi)$ is a bijection of pointed sets.*

PROOF. If $\bar{\alpha}, \bar{\beta}$ are such that $\psi(\bar{\alpha}) \sim \psi(\bar{\beta})$ in $\varprojlim^1 G''_n$, then $\alpha_n = (\beta_n x_n)$ for some choice of x_n in G'_n . Therefore it remains to show only that for each choice of x_n , $(\alpha_n x_n) \sim (\alpha_n)$. The problem is to inductively choose γ_{n+1} such that $(\gamma_{n+1})_n = (\alpha_n^{-1} \gamma_n \alpha_n) x_n$. If $\{G'_n\}$ is a tower of epimorphisms, this is easy. But since $\{G'_n\}$ is Mittag-Leffler, we can assume up to proisomorphism of towers of groups that $\forall k > 0$,

$$\{\text{image } G'_{n+k} \rightarrow G'_n\} = \{\text{image } G'_{n+1} \rightarrow G_n\} = (\text{definition})\{G_n^{(1)}\}.$$

Thus we have a ladder:

$$\begin{array}{ccccccccc} \{1\} & \rightarrow & \{G_n^{(1)}\} & \rightarrow & \{G_n^{(1)}\} & \rightarrow & \{G_n^{(1)}/G_n^{(1)}\} & \rightarrow & \{1\} \\ \downarrow & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \downarrow \\ \{1\} & \rightarrow & \{G'_n\} & \rightarrow & \{G_n\} & \rightarrow & \{G''_n\} & \rightarrow & \{1\} \end{array}$$

Since f' and f are proisomorphisms, f'' is also. Observe that $\{G_n^{(1)}\}$ is a tower of epimorphisms. Since proisomorphisms induce bijections on \varprojlim^1 , this concludes the proof.

The proof of Proposition III is the combination of Lemmas 3.1 and 3.2. For the example, we need a space Y with properties similar to those of ΩS^3 .

LEMMA 3.3. *There exists a finite type CW-space Y such that (a) $Y_0 \cong K(\mathbf{Q}, 2)$, (b) $H^2(Y, \mathbf{Z}) = \mathbf{Z}$ with generator x_2 such that x_2^n is divisible by n^n , and (c) for each prime p and any b in \mathbf{Z}_p , there is $f_b: Y_p \rightarrow Y_p$ such that $f_b^* x_2 = b x_2$.*

PROOF. Construct the Postnikov approximations to Y inductively: $Y^{(2)} = K(\mathbf{Z}, 2)$, and $Y^{(2n)}$ is induced from $Y^{(2n-2)}$ by killing the $\mathbf{Z}/n^n \mathbf{Z}$ reduction of the image of x_2^n of $H^{2n}(K(\mathbf{Z}, 2), \mathbf{Z})$ in $H^{2n}(Y^{(2n-2)}, \mathbf{Z})$. Thus (b) is immediate and (c) follows from the homogeneity of the k -invariants of Y .

EXAMPLE 3.4. Let $X = K(\mathbf{Z}, 2) \times Y$, for Y as 3.3. Then for each prime p , there exists a sequence of positive integers $v(n)$ such that $v(n) \rightarrow \infty$ as $n \rightarrow \infty$ and such that there is a one-to-one correspondence between $SNT(X_p)$ and the quotient of the set of p -adic integers of the form $\sum_1^\infty b_n p^{v(n)}$ under the equivalence relation that $\sum b'_n p^{v(n)} \sim \sum b_n p^{v(n)}$ if and only if there exists a p -local integer B and a p -local unit A such that $A \sum b_n p^{v(n)} + \sum b'_n p^{v(n)} = B$. In particular, W_p in $SNT(X_p)$ is homotopy equivalent to X_p if and only if the corresponding p -adic integer is actually a p -local integer. It also follows that $SNT(X_p)$ is uncountable for each p . We conjecture that this example is valid if Y is replaced with ΩS^3 .

PROOF. By Proposition III, it suffices to consider the tower of groups $H^2(\text{Aut}(X_p^{(2n)}), \mathbf{Q})$, since this is isomorphic to the tower of images of the automorphism groups in the automorphism groups of the rationalization of the $X_p^{(2n)}$. Consider the "components" of $f: X_p^{(2n)} \rightarrow X_p^{(2n)}$: $f_{11}: K(\mathbf{Z}_p, 2)$

$\rightarrow K(\mathbf{Z}_p, 2)$, $f_{12}: K(\mathbf{Z}_p, 2) \rightarrow Y_p^{(2n)}$, $f_{21}: Y_p^{(2n)} \rightarrow K(\mathbf{Z}_p, 2)$ and $f_{22}: Y_p^{(2n)} \rightarrow Y_p^{(2n)}$. Now f_{11} , f_{21} , and f_{22} can induce maps of degree any p -local integer on $H^2(X_p^{(2n)}, \mathbf{Q})$, but the effect of f_{12} is more constrained.

LEMMA 3.5. *Let $g: \mathbf{CP}(n) \rightarrow Y_p$. Then image $H^2(g, \mathbf{Z}_p)$ is divisible by at least $p^{v(n)}$, with $\max_{j \leq n} v_p(j) \leq v(n)$, where $v_p(j)$ is the exponent of the largest power of p dividing j .*

PROOF. This follows immediately from the structure of the cohomology algebras of Y and $\mathbf{CP}(n)$, using 3.3(b). An upper bound for $v(n)$ can be given in terms of the p -primary torsion of the first $2n$ homotopy groups of Y . Any multiple of $p^{v(n)}$ can then be obtained as the degree of f_{12} , if $v(n)$ is taken to be the minimum.

With this characterization of $G_n = H^2(\text{Aut}(X_p^{(2n)}, \mathbf{Q}))$, it is easy to show that each $(\alpha_n) \in \prod_n G_n$ is equivalent to

$$\left\{ \begin{pmatrix} 1 & b'_n p^{v(n)} \\ 0 & 1 \end{pmatrix} \right\}$$

for some choice of (b'_n) . If

$$\left\{ \begin{pmatrix} a_n & r_n \\ c_n & d_n \end{pmatrix} \right\}$$

has the property that

$$\begin{pmatrix} a_{n+1} & r_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_n & r_n \\ c_n & d_n \end{pmatrix}$$

for all $n > 0$ and the r_n , s_n and t_n satisfy the p -divisibility requirements of 3.5, then by recursion, we have that $r_1 = d_1 \Sigma_2^\infty t_n - a_1 \Sigma_2^\infty s_n$ is a valid equation in $\hat{\mathbf{Z}}_p$. Conversely, if there exist p -local integers a_1 , d_1 , and r_1 such that such an expression holds, one can solve for a_n , r_n , c_n and d_n and show that

$$\left\{ \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} \right\} \sim \left\{ \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} \right\}$$

in $\varprojlim G_n$. We remark that the same analysis for \hat{X}_p shows that $SNT(\hat{X}_p) = \{\hat{X}_p\}$ without falling back on Corollary II.

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19174