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SURGERY ON KNOTS

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ABSTRACT. Surgery on two distinct classical knots can create the same 3-manifold.

It is common knowledge, [5], [8], that every closed orientable 3-manifold can be obtained by performing surgery on a link in the 3-sphere S^3. This means that any such 3-manifold bounds a 4-manifold which can be obtained by adding handles, of index two, to the 4-ball; the components of the link are the attaching spheres for the handles, a framing of the link defines the method of handle addition. (A more general process, which consists of removing many copies of $S^1 \times D^2$ and replacing them via arbitrary homeomorphisms of $S^1 \times S^1$, is called 'Dehn surgery'.) Rob Kirby has recently found a tangible equivalence relation, on the class of all framed links, with the property that two such links are equivalent if and only if they yield, after surgery, the same 3-manifold, [4]; (Robert Craggs [2] has a similar theory). Kirby asked if a single equivalence class could contain two different framed knots (links of but one component). The answer, given here, is 'yes'. The question, it should be noted, is a mild version of the 'Property P'-problem, [1], [7], which, in one form, asks whether the surgery function, that maps a framed knot to a 3-manifold, is injective in the sense that only the unknot maps to S^3.

THEOREM. *There is a homology* 3-*sphere M which can be obtained by surgery on either of two distinct knots.*

PROOF. Two presentations of a link L, with components L_1 and L_2, are shown in Figure 1. Each L_i is unknotted and each is null-homotopic in the complement of the other. There is, however, asymmetry between the two components of L; in fact, L was introduced to the author, by Joe Martin, as the simplest example for demonstrating the asymmetry of wrapping numbers.

Let each component of L be allocated the framing -1, and let M be the corresponding 3-manifold produced by surgery. In the diagrams the orientation conventions are as follows: Let N_1 and N_2 be disjoint tubular neighbourhoods of L_1 and L_2, so that M is obtained by removing the interiors of the N_i and sewing back two copies of $S^1 \times D^2$. For each $a \in S^1$, $a \times \partial D^2$ is identified with a curve in ∂N_i which goes once around ∂N_i longitudinally and once meridianally, screwing in a left-handed direction.

Now, because L_2 is unknotted, it is not necessary to use a link of *two* components to construct M. The process of removing L_2 and modifying L_1,

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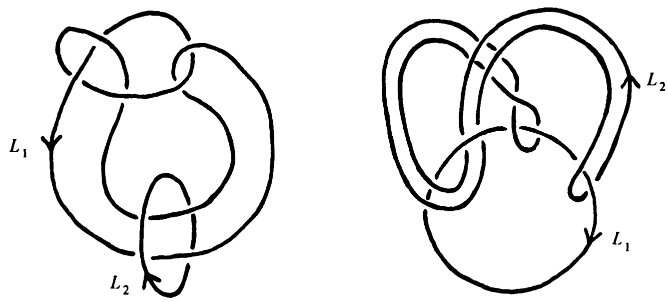

FIGURE 1

so as to create the same 3-manifold, was depicted by Hempel in [3]. Briefly, it is as follows. Cut $S^3 - \mathring{N}_2$ along a disc whose boundary is in N_2, twist one side of the cut through a complete rotation, then glue together again. Assuming that the rotation was in the correct direction, it is now a meridianal curve in ∂N_2 which must needs be identified with $a \times \partial D^2$. This means that N_2 may be replaced whence it came, but the procedure has introduced a pair of cross-overs into L_1, which has now become a knot K_1 (see Figure 2). Hence, surgery on K_1, with framing -1 (because L_2 has linking number zero with L_1), yields M. By reversing the roles of L_1 and L_2, this argument also shows that M can be obtained from the knot K_2.

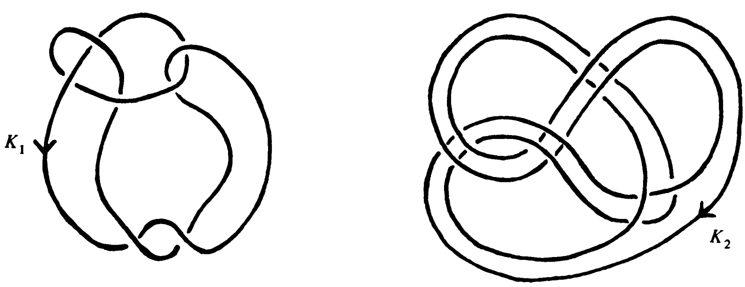

FIGURE 2

The plausible assertion that K_1 and K_2 be distinct can be verified by calculating that their Alexander polynomials are $(1 - 3t + 5t^2 - 3t^3 + t^4)$ and $(1 - t + t^2)$. It is, however, almost obvious from the surgical view of the Alexander polynomial, recently expounded by Dale Rolfsen [6], that the two polynomials must have different degrees.

The Kirby equivalence relation on links is best contemplated in terms of adding 2-handles to the 4-ball. The above example can, in that language, be regarded thus: Consider the 4-manifold obtained by adding a 2-handle to a 4-ball via K_1 with framing -1. Add a 2-handle to the trivial knot with framing -1 (this being a permitted Kirby 'move'). Slide the first handle twice over the second so that the two attaching circles are now linked in link L. Slide the second handle four times over the first, and remove the first handle (which is now trivial) leaving the second handle added via K_2.

Dale Rolfsen reports that he can construct the lens space $L(23, 7)$ by surgery on different knots, though one of his surgeries is a Dehn surgery.

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