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THE EQUIVALENCE OF VARIOUS DEFINITIONS FOR A PROPERLY INFINITE VON NEUMANN ALGEBRA TO BE APPROXIMATELY FINITE DIMENSIONAL

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ABSTRACT. If a properly infinite von Neumann algebra on a separable Hilbert space is approximately finite dimensional with respect to the $*$-ultrastrong topology, that is, if any finite number of elements may be approximated $*$-ultrastrongly by elements of a finite-dimensional sub $*$-algebra, then the algebra may be expressed as the bicommutant of an increasing sequence of factors of type I_{2^n}.

In [2] Murray and von Neumann introduced and proved the equivalence of several definitions that a factor of type II_1 on a separable Hilbert space be approximately finite dimensional. (They used the slightly elliptic term "approximately finite". When a factor of type II_1 came itself to be called "finite", the term "hyperfinite" was introduced as a replacement [1]. This term has since been used to describe certain infinite factors, a situation in which it is not quite appropriate. We use instead the term "approximately finite dimensional".)

In this paper we shall prove that the obvious translations of the definitions of Murray and von Neumann to the infinite case remain equivalent, and that it is not necessary in this case to restrict attention to factors.

LEMMA 1. *Let A be a properly infinite von Neumann algebra, ξ a vector, $\varepsilon > 0$. Then there exists $u \in A$ such that*

(i) $u^*u = 1$ *and* $1 - uu^*$ *is equivalent to* 1;

(ii) $\|(1-u)\xi\| \leqslant \varepsilon$, $\|(1-u)^*\xi\| \leqslant \varepsilon$, $\|(1-uu^*)\xi\| \leqslant \varepsilon$.

PROOF. Choose a projection f equivalent to 1 and such that $\|f\xi\| \leqslant \varepsilon/2$. (If $1 = \sum f_n$ with each f_n equivalent to 1, choose f to be $\sum_{n=n_0}^{\infty} f_n$ with n_0 large.) Choose a projection $e \leqslant f$ such that e and $f-e$ are equivalent. (If $f = \sum_{n=n_0}^{\infty} f_n$ with each f_n equivalent to 1, choose e to be $\sum_{n=n_0}^{\infty} f_{2n}$.) Then e and $f-e$ are both equivalent to f; in particular, there exists v with $v^*v = f$ and $vv^* = f-e$. Set $1 - f + v = u$. Then $(1-u) = f(1-u)f$, whence

$$\|(1-u)\xi\| \leqslant \varepsilon, \qquad \|(1-u)^*\xi\| \leqslant \varepsilon.$$

Moreover, $u^*u = 1$ and $1 - uu^* = e$. Since $\|e\xi\| \leqslant \|f\xi\| \leqslant \varepsilon/2$, we have (ii). Since e is equivalent to f, and hence to 1, we have (i).

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LEMMA 2. *Let A,B be von Neumann algebras, $B \subset A$, ξ a separating vector for A, and $x \in A$. If there exists a sequence (x_n) in B such that $\|(x_n - x)\xi\| \to 0$ and $\|(x_n^* - x^*)\xi\| \to 0$ then $x \in B$.*

PROOF. It follows that $\|[(x_n \pm x_n^*) - (x \pm x^*)]\xi\| \to 0$, and hence it is sufficient to consider the case where the x_n, x are selfadjoint. Since (x_n) may be unbounded, it does not follow that $x_n \to x$ strongly. To avoid this difficulty we use the resolvents. Since $\|(x_n + i)^{-1}\| \leqslant 1$, it follows from the identity

$$(x_n + i)^{-1} - (x + i)^{-1} = (x_n + i)^{-1}(x - x_n)(x + i)^{-1}$$

that $\|[(x_n + i)^{-1} - (x + i)^{-1}]\eta\| \to 0$ where $\eta = (x + i)\xi$. Since ξ is cyclic for A' and $x + i$ is invertible in A, it follows that η is also cyclic for A'. It now follows easily that $(x_n + i)^{-1} \to (x + i)^{-1}$ strongly.

THEOREM 3. *Let A be a properly infinite von Neumann algebra acting on a separable Hilbert space. Then the following three conditions are equivalent.*

(i) *For any $x_1, \ldots, x_n \in A$ and any $*$-ultrastrong neighbourhood V of 0 in A there exist a finite-dimensional sub $*$-algebra N of A and $y_1, \ldots, y_n \in N$ such that $x_i - y_i \in V$, $i = 1, \ldots, n$.*

(ii) *Condition (i) holds and N may be chosen to be a factor of type I_{2^k}.*

(iii) *There exists an increasing sequence of factors N_k of type I_{2^k} such that $A = (\bigcup N_k)''$.*

PROOF. (i) $\Rightarrow$ (ii). Let $x_1, \ldots, x_n \in A$. Let V be a $*$-ultrastrong neighbourhood of 0 in A. We may suppose, replacing the Hilbert space on which A acts by its tensor product with another Hilbert space, and A by $A \otimes 1$, that V is determined by a single vector ξ, that is, that

$$V = \{a \in A; \|a\xi\| \leqslant 1, \|a^*\xi\| \leqslant 1\}.$$

Replacing V by a smaller neighbourhood, we may suppose that ξ is separating for A.

By (i), there exist a finite-dimensional sub $*$-algebra M of A, and $z_1, \ldots, z_n \in M$ such that $x_i - z_i \in 3^{-1}V$, $i = 1, \ldots, n$.

Since ξ is separating, there exists $\delta > 0$ such that if $u^*u = 1$, $u - 1 \in \delta V$, $u^* - 1 \in \delta V$ then $uz_iu^* - z_i \in 3^{-1}V$, $i = 1, \ldots, n$. Choose u as given by Lemma 2 with $\varepsilon = \min(\delta, (3 \sup\|z_i\|)^{-1})$; then $u^*u = 1$, $1 - uu^*$ is equivalent to 1, and

$$uz_iu^* - z_i \in 3^{-1}V, \qquad i = 1, \ldots, n,$$
$$1 - uu^* \in (3 \sup\|z_i\|)^{-1}V.$$

Write M as a direct sum of factors N_r, of type I_{n_r}, with unit $e^{(r)}$, $r = 1, \ldots, R$. Write

$$z_i = \sum_{r,p,q} z_{i,pq}^{(r)} e_{pq}^{(r)},$$

where $(e_{pq}^{(r)})$ is a complete system of matrix units for N_r, and $z_{i,pq}^{(r)} \in \mathbf{C}$. Choose $k = 1, 2, \ldots$ such that $2^k \geqslant t = \sum_{r=1}^{R} n_r$. Partition $1 - uu^*$ into 2^k mutually equivalent projections. Denote the first t of these by $\varepsilon_{pp}^{(r)}$, $p = 1, \ldots, n_r$, $r = 1, \ldots, R$.

For each $r = 1, \ldots, R$, extend $(\varepsilon_{pp}^{(r)})_{p=1,\ldots,n_r}$ to a system of matrix units $(\varepsilon_{pq}^{(r)})_{p,q=1,\ldots,n_r}$. Set

$$(ue_{pq}^{(r)}u^* + \varepsilon_{pq}^{(r)}) = (f_{pq}^{(r)});$$

then $(f_{pq}^{(r)})$ is a system of matrix units. Set

$$\sum_{r,p,q} z_{i,pq}^{(r)} f_{pq}^{(r)} = y_i, \qquad i = 1, \ldots, n.$$

Then for each $i = 1, \ldots, n$,

$$\|(uz_iu^* - y_i)\xi\| = \left\| \sum_{r,p,q} z_{i,pq}^{(r)} \varepsilon_{pq}^{(r)} \xi \right\| \leqslant \|z_i\| \left\| \sum_{r,p} \varepsilon_{pp}^{(r)} \xi \right\|$$
$$\leqslant \|z_i\| \, \|(1 - uu^*)\xi\| \leqslant 3^{-1};$$
$$\|(uz_i^*u^* - y_i)\xi\| \leqslant \|z_i^*\| \, \|(1 - uu^*)\xi\| \leqslant 3^{-1}; \quad uz_iu^* - y_i \in 3^{-1}V.$$

Hence,

$$x_i - y_i = (x_i - z_i) + (z_i - uz_iu^*) + (uz_iu^* - y_i)$$
$$\in 3^{-1}V + 3^{-1}V + 3^{-1}V = V, \qquad i = 1, \ldots, n.$$

Since the projections obtained from the partition of $1 - uu^*$ are all equivalent to 1, the projections $f_{pp}^{(r)}$ and the remaining $2^k - t$ projections are mutually equivalent. Hence the systems of matrix units $(f_{pq}^{(r)})$, $r = 1, \ldots, R$, can be extended to a common system of matrix units, the sum of whose projections is 1. The linear span of these matrix units is a factor of type I_{2^k}, which is contained in A and contains the above y_i, and therefore satisfies the requirements for N.

(ii) $\Rightarrow$ (iii). The argument is basically the same as in the case that A is of type II_1. It is enough to construct an increasing sequence of factors N_{k_n} of type $\mathrm{I}_{2^{k_n}}$ such that $A = (\bigcup N_{k_n})''$, and we shall do this inductively.

Choose a sequence $(x_1, x_2, \ldots)$ $*$-ultrastrongly dense in A. We may suppose that there exists a vector ξ separating for A. Fix $n = 1, 2, \ldots$, and assume that there is a factor $N_{k_n} \subset A$ of type $\mathrm{I}_{2^{k_n}}$ such that for suitable $y_1^{(n)}, \ldots, y_n^{(n)}$ in N_{k_n},

$$\|(x_i - y_i^{(n)})\xi\| \leqslant n^{-1}, \quad \|(x_i - y_i^{(n)})^*\xi\| \leqslant n^{-1}, \qquad i = 1, \ldots, n.$$

Choose a complete system of matrix units (e_{pq}) for N_{k_n}. Then $e_{11}Ae_{11}$ satisfies (ii) in place of A, as e_{11} is equivalent to 1 and so $e_{11}Ae_{11}$ is isomorphic to A. In particular, there exists a factor $N \subset e_{11}Ae_{11}$ of type I_{2^k} (with unit e_{11}) such that for suitable $y_{1,pq}, \ldots, y_{n+1,pq} \in N$,

$$\|(e_{1p}x_ie_{q1} - y_{i,pq})e_{1q}\xi\| \leqslant (n+1)^{-1}k_n^{-2},$$
$$\|(e_{1p}x_ie_{q1} - y_{i,pq})^*e_{1p}\xi\| \leqslant (n+1)^{-1}k_n^{-2},$$
$$i = 1, \ldots, n+1, \quad p, q = 1, \ldots, k_n.$$

Set

$$\sum_{p,q} e_{p1}y_{i,pq}e_{1q} = y_i^{(n+1)}, \qquad i = 1, \ldots, n+1.$$

Then for each $i = 1, \ldots, n+1$, $y_i^{(n+1)}$ is in the algebra $N_{k_{n+1}}$ generated by N_{k_n} and N, a factor of type $\mathrm{I}_{2^{k_n+k}}$, and for each $i = 1, \ldots, n$,

$$\begin{aligned}\|(x_i - y_i^{(n+1)})\xi\| &= \left\|\sum_{p,q} e_{p1}(e_{1p}x_ie_{q1} - y_{i,pq})e_{1q}\xi\right\| \\ &\leqslant \sum_{p,q}(n+1)^{-1}k_n^{-2} = (n+1)^{-1},\\ \|(x_i - y_i^{(n+1)})^*\xi\| &= \left\|\sum_{p,q} e_{q1}(e_{1p}x_ie_{q1} - y_{i,pq})^*e_{1p}\xi\right\| \\ &\leqslant \sum_{p,q}(n+1)^{-1}k_n^{-2} = (n+1)^{-1}.\end{aligned}$$

This shows that there exist an increasing sequence of factors $N_{k_n} \subset A$ of type $\mathrm{I}_{2^{k_n}}$ and a separating vector ξ for A such that the hypotheses of Lemma 2 are satisfied for all $x \in A$ with $B = (\bigcup N_{k_n})''$. Hence $A = (\bigcup N_{k_n})''$.

(iii) $\Rightarrow$ (i). This follows from the bicommutant theorem.

REFERENCES

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*, 2ième éd., Gauthier-Villars, Paris, 1969. MR **50** #5482.

2. F. J. Murray and J. von Neumann, *On rings of operators*. IV, Ann. of Math. (2) **44** (1943), 716–808. MR **5**, 101.

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