CHARACTERISTIC NUMBERS FOR UNORIENTED SINGULAR G -BORDISM

S. S. KHARE AND B. L. SHARMA

ABSTRACT. We develop the notion of characteristic numbers for unoriented singular G -manifolds in a G -space, G being a finite group, and prove their invariance with respect to unoriented singular G -bordism.

Thom [5] gave the notion of Stiefel Whitney numbers and Pontrjagin numbers of a manifold M^n and proved its invariance with respect to bordism. Chung N. Lee and Arthur Wasserman [4] developed these notions for G -manifolds. In this note we have developed these notions for unoriented singular principal G -manifolds in a G -space, G being a finite group, and proved their invariance with regard to unoriented singular G -bordism.

1. Characteristic numbers. Let X be a finite CW-complex with free action of G , G being a finite group, and X/G be again a finite CW-complex. Let h^* be an equivariant cohomology theory and h_* be the associated equivariant homology theory [1]. Let $h^* = H^* \circ A$ and $h_* = H_* \circ A$, where A is a functor from the category of G -spaces and equivariant maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle , \rangle : h^*(X; G) \otimes_{H^*(pt.)} h_*(X; G) \rightarrow H_*(pt.)$$

be the Kronecker pairing.

Let us assign to each compact G -manifold W , a class

$$[W, \partial W] \in h_*(W, \partial W; G)$$

such that

$$(a) [W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1, \partial W_1] + [W_2, \partial W_2],$$

$$(b) \partial_*[W, \partial W] = [\partial W].$$

Suppose $[M^n, f; G]$ is an element of unoriented bordism group $\mathfrak{N}_n(\check{X}; G)$ [3] and $x \in h^*(B(O, G)_n; G)$, $B(O, G)_n$ being the classifying space for G -vector bundles of dimension n . Then the x -characteristic number of the map $f: M^n \rightarrow X$ associated with an element $a^m \in h^m(X; G)$ is defined to be

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$\langle \tau_{M^n}^*(x) f^*(a^m), [M] \rangle$, where $\tau_{M^n}: M^n \rightarrow B(O, G)_n$ is the tangent map.

In particular, let the equivariant cohomology h^* be given by $h^*(X; G) = H^*((E_G \times X)/G; \mathbf{Z}_2)$ and h_* be the associated equivariant homology, i.e. $h_*(X; G) = H_*((E_G \times X)/G; \mathbf{Z}_2)$, where the action of G on $E_G \times X$ is given by $g(e, x) = (ge, gx)$, E_G being the total space of the universal G -bundle. Consider the map $q: X/G \rightarrow (E_G \times X)/G$ given by $q([x]) = [\bar{\alpha}(x), x]$, where $\bar{\alpha}$ is the map given by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\bar{\alpha}} & E_G \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{\alpha} & BG \end{array}$$

The map q is homotopy equivalence. Thus

$$h^*(X; G) \overset{q^*}{\approx} H^*(X/G; \mathbf{Z}_2) \quad \text{and} \quad h_*(X; G) \overset{q_*^{-1}}{\approx} H_*(X/G; \mathbf{Z}_2).$$

Therefore $h_*(M^n; G) \approx H_*(M^n/G; \mathbf{Z}_2)$ has a topological class, say σ_n , in dimension n .

2. Invariance of characteristic numbers. Throughout the section we will be considering equivariant cohomology h^* to be

$$h^*(X; G) = H^*((E_G \times X)/G; \mathbf{Z}_2)$$

and equivariant homology h_* to be

$$h_*(X; G) = H_*((E_G \times X)/G; \mathbf{Z}_2).$$

THEOREM 2.1. *If $[M^n, f; G] \in \mathfrak{R}_n(X; G)$ is zero then all the x -characteristic numbers of the map $f: M^n \rightarrow X$ associated with every $a^m \in h^m(X; G)$ are zero.*

PROOF. Since $[M^n, f; G] \in \mathfrak{R}_n(X; G)$ is zero, $\exists$ an $(n+1)$ -dimensional compact principal G -manifold W^{n+1} and an equivariant map $F: W^{n+1} \rightarrow X$ with $\partial W^{n+1} = M^n$ and $F/M^n = f$. Let $\omega_{n+1} \in h_{n+1}(W^{n+1}, \partial W^{n+1}; G)$ be the topological class of W^{n+1} . Then $\partial_*(\omega_{n+1}) = \sigma_n$. We have the following commutative diagram:

$$\begin{array}{ccc} h^*(B(O, G)_n; G) & \xrightarrow{\tau_{M^n}^*} & h^*(M^n; G) \\ \uparrow j^* & & \uparrow i^* \\ h^*(B(O, G)_{n+1}; G) & \xrightarrow{\tau_{W^{n+1}}^*} & h^*(W^{n+1}; G), \end{array}$$

where $j: B(O, G)_n \rightarrow B(O, G)_{n+1}$ is the map classifying $\mu_n \oplus 1$, $\mu_n \rightarrow B(O, G)_n$ being the universal G -vector bundle. Also we have

$$\begin{aligned} h^*(B(O, G)_n; G) &= H^*((E_G \times B(O, G)_n)/G; \mathbf{Z}_2) \\ &= H^*(BG \times BO_n; \mathbf{Z}_2) \quad [6] \\ &= H^*(BG; \mathbf{Z}_2) \otimes H^*(BO_n; \mathbf{Z}_2) \end{aligned}$$

and

$$h_*(B(O, G)_n; G) = H_*(BG; \mathbb{Z}_2) \otimes H_*(BO_n; \mathbb{Z}_2).$$

Thus the map j^* is a surjection. Therefore for every $x \in h^*(B(O, G)_n; G)$, $\exists y \in h^*(B(O, G)_{n+1}; G)$ such that $j^*(y) = x$. Therefore

$$\begin{aligned} \langle \tau_{M^n}^*(x) f^*(a^m), \sigma_n \rangle &= \langle \tau_{M^n j^*}^*(y) f^*(a^m), \sigma_n \rangle = \langle i^* \tau_{W^{n+1}}^*(y) i^* F^*(a^m), \sigma_n \rangle \\ &= \langle \tau_{W^{n+1}}^*(y) F^*(a^m), i_* \partial_*(\omega_{n+1}) \rangle = 0. \end{aligned}$$

This completes the proof of the theorem.

Consider now the map $\mu: \mathfrak{N}_*(X; G) \rightarrow h_*(X; G)$ defined as $\mu([M^n, f; G]) = q_* \bar{f}_*(\bar{\sigma}_n)$, where $\bar{f}$ is the map given by the following commutative diagram:

$$\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ M^n/G & \xrightarrow{\bar{f}} & X/G \end{array}$$

$\bar{\sigma}_n \in H_n(M^n/G; \mathbb{Z}_2)$ being the fundamental class and let $\bar{\mu}: \mathfrak{N}_*(X/G) \rightarrow H_*(X/G; \mathbb{Z}_2)$ be the map defined by $\bar{\mu}([N^n, g]) = g_*(\bar{\sigma}'_n)$, where $\bar{\sigma}'_n \in H_n(N^n; \mathbb{Z}_2)$ is the fundamental class. Suppose $\phi_*: \mathfrak{N}_*(X; G) \rightarrow \mathfrak{N}_*(X/G)$ is the isomorphism [3] defined as $\phi_*([M^n, f; G]) = [M^n/G, \bar{f}]$. Then $\mu = q_* \bar{\mu} \phi_*$ and, therefore, μ is an epimorphism, since $\bar{\mu}$ is so [2]. For every $a \in h_*(X; G)$, we select $[M^n, f; G] \in \mathfrak{N}_*(X; G)$ such that $\mu([M^n, f; G]) = a$. We define the $\mathfrak{N}$ -module structure on $h_*(X; G)$ by

$$[V^m]a = \mu[M^n \times V^m, f'; G],$$

for every $[V^m] \in \mathfrak{N}$, where the action of G on $M^n \times V^m$ is defined as $g(x, y) = (gx, y)$ and $f': M^n \times V^m \rightarrow X$ is defined as $f'(x, y) = f(x)$. Thus $h_*(X; G) \otimes \mathfrak{N}$ is a $\mathfrak{N}$ -module. Let $\{C_{n,i}\}$ be the additive base of $h_*(X; G)$. Let $[M_i^n, f_i; G] \in \mathfrak{N}_*(X; G)$ with $\mu([M_i^n, f_i; G]) = C_{n,i}$. We define $h: h_*(X; G) \otimes \mathfrak{N} \rightarrow \mathfrak{N}_*(X; G)$ by $h(C_{n,i} \otimes 1) = [M_i^n, f_i; G]$.

THEOREM 2.2. *The map $h: h_*(X; G) \otimes \mathfrak{N} \rightarrow \mathfrak{N}_*(X; G)$, defined as above is an isomorphism.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccc} h_*(X; G) \otimes \mathfrak{N} & \xrightarrow{h} & \mathfrak{N}_*(X; G) \\ \downarrow q_*^{-1} \otimes 1_{\mathfrak{N}} & & \downarrow \phi_* \\ H_*(X/G; \mathbb{Z}_2) \otimes \mathfrak{N} & \xrightarrow{\bar{h}} & \mathfrak{N}_*(X/G) \end{array}$$

where $\bar{h}: H_*(X/G; \mathbb{Z}_2) \otimes \mathfrak{N} \rightarrow \mathfrak{N}_*(X/G)$ is defined as $\bar{h}(\bar{C}_{n,i} \otimes 1) = [M_i^n/G; \bar{f}_i]$, where $\bar{C}_{n,i} = q_*^{-1}(C_{n,i})$. We already know that $\bar{h}$ is an isomorphism [2] and, therefore, so is h .

The above theorem gives the converse of Theorem 2.1 given as below.

THEOREM 2.3. *If all the characteristic numbers of an element $[M^n, f; G]$ $\in \mathfrak{N}_*(X; G)$ are zero, then $[M^n, f; G] = 0$.*

PROOF. Let $\mu([M^n, f; G]) = C_n \in h_*(X; G)$ and $q_*^{-1}(C_n) = \bar{C}_n \in H_*(X/G; \mathbb{Z}_2)$. Therefore $\bar{f}_*(\bar{\sigma}_n) = \bar{C}_n$. Suppose $\{C_{n,i}\}_{i \in I}$ is an additive base of $h_n(X; G)$ and $C^{n,j} \in h^n(X; G)$ is the cohomology class dual to $C_{n,i}$ in the sense $\langle \bar{C}^{n,j}, \bar{C}_{n,i} \rangle = \delta_{ij}$, where $q_*^{-1}(C_{n,i}) = \bar{C}_{n,i}$ and $q^*(C^{n,j}) = \bar{C}^{n,j}$. Let $C_n = \sum_{i \in S} \pm C_{n,i}$, S being a finite subset of I . Then if $C^n = \sum_{i \in S} \pm C^{n,i}$, by hypothesis the x -characteristic number of $[M^n, f; G]$ associated with $C^n \in h^n(X; G)$ is zero, that means taking x to be unit class of $h^*(B(O, G)_n; G)$,

$$\langle f^*(C^n), [M] \rangle = 0, \quad \text{or} \quad \langle f^*(C^n), q_*[\bar{\sigma}_n] \rangle = 0,$$

or

$$\langle (q^*)^{-1}(\bar{C}^n), f_* q_*[\bar{\sigma}_n] \rangle = 0, \quad \text{or} \quad \langle (q^*)^{-1}(\bar{C}^n), q_* \bar{f}_*[\bar{\sigma}_n] \rangle = 0,$$

by the following commutative diagram

$$\begin{array}{ccc} h_n(M^n; G) & \xrightarrow{f_*} & h_n(X; G) \\ \downarrow q_*^{-1} & & \downarrow q_*^{-1} \\ H_n(M^n/G; \mathbb{Z}_2) & \xrightarrow{\bar{f}_*} & H_n(X/G; \mathbb{Z}_2) \end{array}$$

Therefore $\langle (q^*)^{-1}(\bar{C}^n), q_*[\bar{C}_n] \rangle = 0$, which implies that $\langle \bar{C}^n, \bar{C}_n \rangle = 0$, showing that $\bar{C}_n = 0$. Also it is easy to see that $h(C_n \otimes 1) = [M^n, f; G]$. Since h is an isomorphism and $C_n = 0$, $[M^n, f; G] = 0$, which completes the proof of the theorem.

Theorems 2.1 and 2.3 give the invariance of characteristic numbers with regard to unoriented singular principal G -bordism.

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DEPARTMENT OF MATHEMATICS, NORTH-EASTERN HILL UNIVERSITY, RITA VILLA, LOWER LACHAUMIERE, SHILLONG-793 001, MEGHALAYA, INDIA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALLAHABAD, ALLAHABAD (U.P.), INDIA