REPRESENTING HOMOLOGY CLASSES OF CLOSED ORIENTABLE SURFACES

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ABSTRACT. We prove that the primitive elements of the first homology group of a genus g torus can be represented by a simple closed curve.

Let T be a closed orientable surface with g handles ($g \geq 0$). The first homology group, $H_1(T)$, is the free abelian group on $2g$ generators. In general, if an element of $H_1(T)$ is represented by a closed curve, the curve must cross itself. However, we prove that if the element is primitive, then it can be represented by a simple closed curve. Primitive means not divisible by an integer greater than one. Our proof is accomplished through use of the "twist" homeomorphism described by Lickorish [1].

Let α_k and β_k be the standard oriented simple closed curves around the k th handle of T for $1 \leq k \leq g$. So each α_i or β_j represents a basis element of $H_1(T)$. For $1 \leq k < g$, let γ_k be a simple closed curve looping the k th and $(k+1)$ st handles of T . The Lickorish "twist" homeomorphism about a simple closed curve J is defined by cutting T along J , giving a full twist to one edge, and then gluing T back together along J . Twists about α_k , β_k , and γ_k induce automorphisms on $H_1(T)$ carrying $(\dots, a_k, b_k, a_{k+1}, b_{k+1}, \dots)$ to $(\dots, a_k \pm b_k, b_k, a_{k+1}, b_{k+1}, \dots)$, $(\dots, a_k, b_k \pm a_k, a_{k+1}, b_{k+1}, \dots)$, and $(\dots, a_k \pm b_k \pm b_{k+1}, b_k, a_{k+1} \pm b_k \pm b_{k+1}, b_{k+1}, \dots)$ respectively. The " $\pm$ " signs appear since there is always a choice of two ways of twisting.

Now let $(a_1, b_1, a_2, b_2, \dots)$ be a fixed element of $H_1(T)$. By several appropriate iterations of twists about α_k 's and β_k 's, and using the Euclidean algorithm, we get a homeomorphism which induces a map carrying $(a_1, b_1, a_2, b_2, \dots)$ to $(d_1, 0, d_2, 0, \dots)$, where $d_i = \text{g.c.d.}\{a_i, b_i\}$. By following this by further twists about α_k 's, β_k 's, and γ_k 's, we can produce a self-homeomorphism of T which induces a map carrying $(a_1, b_1, a_2, b_2, \dots)$ to $(d, 0, 0, 0, \dots)$, where

$$d = \text{g.c.d.}\{d_1, d_2, \dots\} = \text{g.c.d.}\{a_1, b_1, a_2, b_2, \dots\}.$$

For $d = \pm 1$, $(d, 0, 0, 0, \dots)$ is represented by the simple closed curve $\pm \alpha_1$, and by applying the inverse of the homeomorphism we have constructed to α_1 we get the simple closed curve we desire.

The converse is true, namely simple closed curves represent primitive (or trivial) elements. The following argument is due to Hans Samelson. A

Received by the editors March 25, 1975.

AMS (MOS) subject classifications (1970). Primary 57A05.

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nonbounding simple closed curve J (i.e., not representing a trivial element) is always intersected by some other closed curve exactly once. Just take a small arc crossing J once and connect the end points in the arcwise connected set $T - J$. But if J represents a homology class divisible by m , then its intersection number with any other closed curve is a multiple of m . So J represents a primitive element.

REFERENCES

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